KEENAN
ASL 1989

Sortal Reducibility Theorem

Def Given a non-empty universe E, F from P(E) into [P(E) --> 2] is sortally reducible iff for some boolean function h of two variables, F(p)(q) = F(E)(h(p,q)), all p,q $\in$ P(E). F is called inherently sortal if F is not sortally reducible.

Theorem (Sortal Reducibility)

For F conservative, F is sortally reducible iff F is intersective or F is co-intersective

proof

A. <==

if F is intersective then F(p)(q) = F(E)(p $\cap$ q) since p $\cap$ q = E $\cap$ (p $\cap$ q), all p,q.

if F is co-intersective then F(p)(q) = F(E)(-p v q) since E - (-p v q) = E $\cap$ (-(-p v q)) = p $\cap$ -q = p - q

B. ==>

The 16 boolean functions in two variables are:

h_1(p)(q)	=	E	g_1(p)(q)	=	$\emptyset$
h_2	=	p	g_2	=	$\neg$p
h_3	=	q	g_3	=	$\neg$q
h_4	=	p $\cap$ q	g_4	=	$\neg$p v $\neg$q
h_5	=	p $\cap$ $\neg$q	g_5	=	$\neg$p v q
h_6	=	$\neg$p $\cap$ q	g_6	=	p v $\neg$q
h_7	=	$\neg$p $\cap$ $\neg$q	g_7	=	p v q
h_8	=	(p - q) v (q - p)	g_8	=	(p $\cap$ q) v ($\neg$p $\cap$ $\neg$q)

1. Suppose F reduces via h_1. That is, F(p)(q) = F(E)(E), all p,q. Then F is constant, so trivial and thus intersective. Ditto if F reduces via g_1.

2. Suppose F(p)(q) = F(E)(h_2(p,q). Then F(p)(q) = F(E)(p) = F(E)(E), whence F is trivial and thus intersective. Ditto if F(p)(q) = F(E)(g_2(p,q)) = F(E)($\neg$p) = F(E, g_2(E,-p)) = F(E, $\emptyset$)

3. Suppose F(p)(q) = F(E)(h_3(p,q)) = F(E)(q). Then for all p,p' F(p)(q) = F(p')(q). But by CONS, F($\emptyset$)(q) = F($\emptyset$)($\emptyset$ $\cap$ q) = F($\emptyset$)($\emptyset$) = F(E)($\emptyset$). Thus for all p,q F(p)(q) = F(E)($\emptyset$), so again F is trivial and thus intersective. Ditto when F(p)(q) = F(E)(g_3(p,q)) = f(E)($\neg$q)

4. If F(p)(q) = F(E)(h_4(p,q)) = F(E)(p $\cap$ q) then F is intersective. Ditto if F(p)(q) = F(E)(g_4(p,q)) = F(E)($\neg$(p $\cap$ q))

5. Suppose $F(p)(q) = F(E)(h_5(p,q)) = F(E)(p \cap \neg q)$. Then F is co-intersective. Ditto if $F(p)(q) = F(E)(g_5(p,q)) = F(E)(\neg p \vee q)$.

6. Suppose $F(p)(q) = F(E)(h_6(p,q)) = F(E)(\neg p \cap q)$. Then by CONS $F(p)(q) = F(p)(p \cap q) = F(E)(\neg p \cap (p \cap q)) = F(E)(\emptyset)$, so F is constant and thus intersective.

   Suppose $F(p)(q) = F(E)(g_6(p,q)) = F(E)(p \vee \neg q)$. Then $F(p)(q) = F(p)(p \cap q) = F(E)(p \vee \neg(p \cap q)) = F(E)(E)$, so F is trivial and thus intersective

7. Suppose $F(p)(q) = F(E)(h_7(p,q)) = F(E)(\neg p \cap \neg q)$. Then $F(p)(q) = F(E)(\neg p \cap \neg q) = F(E)(\neg E \cap \neg(\neg p \cap \neg q)) = F(E)(\emptyset)$, so F is trivial and thus intersective. Equally F is trivial if $F(p)(q) = F(E)(g_7(p,q)) = F(E)(p \vee q) = F(E)(E \vee (p \vee q)) = F(E)(E)$.

8. Suppose $F(p)(q) = F(E)(h_8(p,q)) = F(E)((p - q) \vee (q - p))$. Then $F(p)(q) = F(p)(p \cap q) = F(E)(p - (p \cap q)) \vee ((p \cap q) - p) = F(E)(p - q)$, so F is co-intersective.

   And if $F(p)(q) = F(E)(g_8(p,q)) = F(E)((p \cap q) \vee (\neg p \cap \neg q))$ then $F(p)(q) = F(p)(p \cap q) = F(E)((p \cap (p \cap q)) \vee (\neg p \cap \neg(p \cap q))) = F(E)((p \cap q) \vee \neg p)) = F(E)(\neg p \vee q)$, so F is co-intersective.

   This exhausts the cases, proving the theorem. ♥♥

Generalized (Quantifier) Prefix Theorem (GPT)

preliminaries

Given a non-empty universe E, write R_n for $P(E^n)$, the set of n-ary relations over E. The set $[R_n \to 2]$ of functions from R_n into {0,1} is the set of Generalized n-ary quantifiers (over E).

So the generalized quantifiers to date are the generalized unary quantifiers.

def Where F is a generalized unary quantifier (over E) we extend the domain of F to include all n+1 ary relations R by setting

$$F(R) = \{\langle a_1,\ldots,a_n\rangle \mid F(\{b \mid \langle a_1,\ldots,a_n,b\rangle \in R\}) = 1\}$$

So F sends each n+1 ary relation to an n-ary one, and the domain of F is the set of n+1 ary relations, all $n \geq 0$.

def A sequence $\langle F_1, \ldots F_n \rangle$ of generalized unary quantifiers induces an n-ary quantifier $F = (F_1, \ldots, F_n)$ given by

$$F(R) = F_1(F_2(\ldots(F_n(R))\ldots))$$

def An n-ary quantifier H is (unary) reducible (or Fregean) iff H is induced by some sequence $\langle H_1, \ldots, H_n \rangle$ of unary quantifiers.

Major Queries

1. When do distinct n-ary sequences of unary quantifiers induce the same n-ary quantifier?
2. Characterize the n-ary quantifiers which are Fregean (and those which are not). Is there any reason why, in ordinary mathematical logics, we should have chosen Fregean analyses of multiply quantified formulas?

Query 1

def The trivial n-ary quantifiers are 0_n, which sends all n-ary R to 0, and 1_n, which sends all n-ary R to 1

def Given F n-ary, write $F^{\neg}$, the post-complement of F, for that n-ary quantifier given by

$$(F^{\neg})(R) = F(\neg R)$$

($\neg R$ is just the set of n-tuples not in R, of course)

Lemma (TRIV)

a. if F is a non-trivial as a unary quantifier then the range of $F \restriction R_{n+1}$ is R_n

[if $F(\emptyset) = 0$ then for some non-empty p, $F(p) = 1$ and so for all n-ary R, $F(R \times p) = R$.
if $F(\emptyset) = 1$ then for some non-empty p, $F(p) = 0$ and so for all n-ary R, $F(\neg R \times p) = R$]

b. $(F_1, \ldots, F_n)$ is trivial iff for some $1 \leq i \leq n$, F_i is trivial

<== Say $F_i = 0$. Then for all n-ary R, S

$(F_1, \ldots, F_n)(R) = (F_1, \ldots, F_{i-1})(\emptyset) = (F_1, \ldots, F_n)(S)$, so $(F_1, \ldots, F_n)$ is trivial.

==> follows by induction from a. above. ♥

KEENAN
ASL 1989

Theorem (Generalized Quantifier Prefix Theorem)

Let **F** = $(F_1, \ldots, F_{n+1})$ and **G** = $(G_1, \ldots, G_{n+1})$ be non-trivial n+1 ary reducible quantifiers as indicated. Then

F = **G** iff either (a) or (b) below holds:

(a) $(F_1, \ldots, F_n) = (G_1, \ldots, G_n)$ & $F_{n+1} = G_{n+1}$

(b) $(G_1, \ldots, G_n) = (F_1, \ldots, F_n)\urcorner$ & $G_{n+1} = \neg F_{n+1}$

proof

i. The right to left direction is obvious on a moment's reflection

ii. The left to right direction is given by a sequence of lemmas below. We write F^* for $(F_1, \ldots, F_n)$ and ditto for G^*

lemma 1: For F and G as in the theorem, assume **F** = **G**. Then,

if $F_{n+1}(\emptyset) = G_{n+1}(\emptyset) = 0$ then condition (a) obtains

pf: A. $F^* = G^*$. Suppose first that there is a property s such that $F_{n+1}(s) = G_{n+1}(s) = 1$. So $s \neq \emptyset$. Then for all n-ary relations R,

$F^*(R) = (F^*, F_{n+1})(R \times s) = (G^*, G_{n+1})(R \times s) = G^*(R)$, so in this case $F^* = G^*$. Moreover, this exhausts the cases.

Viz, suppose there is no s such that $F_{n+1}(s) = G_{n+1}(s) = 1$. By the non-trivialty of F_{n+1} (TRIV.b) let q such that $F_{n+1}(q) = 1$. So $q \neq \emptyset$ and $G_{n+1}(q) = 0$. Then, for all n-ary relations R,

$F^*(R) = (F^*, F_{n+1})(R \times q) = (G^*, G_{n+1})(R \times q) = G^*(\emptyset)$, so F^* is constant, contradicting that it is non-trivial.

B. Given (A), show $F_{n+1} = G_{n+1}$. Suppose, leading to a contradiction that $F_{n+1}(s) \neq G_{n+1}(s)$. Say that $F_{n+1}(s) = 1$ and $G_{n+1}(s) = 0$. Then from the non-triviality of F^*, (TRIV.b), let p such that $F^*(p) \neq F^*(\emptyset)$. Then,

$$F^*(p) = (F^*, F_{n+1})(p \times s) = (G^*, G_{n+1})(p \times s) = G^*(\emptyset)$$

$$= F^*(\emptyset) \quad \text{(from A. above)}$$

a contradiction, proving the lemma. ♥

<u>lemma</u> 2 For **F** = **G** as above, if $F_{n+1}(\emptyset) = G_{n+1}(\emptyset) = 1$ then, once again, condition (a) obtains.

pf: from (i) observe that

$$F = (F^{*}, F_{n+1}) \quad = \quad (F^{*\neg}, \ \neg F_{n+1}) \quad \text{and}$$

$$G = (G^{*}, F_{n+1}) \quad = \quad (G^{*\neg}, \ \neg G_{n+1})$$

Applying lemma 1 to the righthand sides here we infer $\neg F_{n+1} = \neg G_{n+1}$, whence $F_{n+1} = G_{n+1}$; and equally $F^{*\neg} = G^{*\neg}$, whence $F^{*} = G^{*}$ (post-complement is injective).

<u>lemma</u> 3 Let **F** = **G** and $F_{n+1}(\emptyset) \neq G_{n+1}(\emptyset)$. Say $F_{n+1}(\emptyset) = 1$ and $G_{n+1}(\emptyset) = 0$.

Then $F \ = \ (F^{*}, F_{n+1}) = (F^{*\neg}, \ \neg F_{n+1}) \ = \ (G^{*}, G_{n+1})$

Applying lemma 1 then $G^{*} = F^{*\neg}$ and $G_{n+1} = \neg F_{n+1}$, satisfying condition (b) of the theorem. ♥

As the three lemmas cover all the cases we have that **F** = **G** implies that condition (a) or condition (b) obtains, completing the proof of the theorem. ♥♥

<u>Discussion of the Generalized Prefix Theorem</u>

A. <u>Linguistic</u>

The following (a,b) pairs of English sentences illustrate the equivalences given by condition (b) of the theorem. E.g. they are of the form:

a. NP_1 V NP_2 b. NP_1-neg V neg-NP_2

a. Each boy read fewer than six plays (over the vacation)
b. NO boy read six or more plays (over the vacation)

a. Both John and Bill read at least as many plays as novels
b. Neither John nor Bill read more novels than plays

a. More than half the boys answered no questions correctly
b. Less than half the boys answered any questions correctly

a. Exactly half the boys read more plays than poems
b. Exactly half the boys read at least as many poems as plays

a. All but six students read more plays than novels
b. Exactly six students read as many novels as plays

a. None of John's cats caught more than two mice
b. Each of John's cats caught at most two mice

a. Not every student knows as many male as female teachers
b. Some student knows more female than male teachers

a. Proportionately more boys than girls read at most six plays
b. Proportionaltely more girls than boys read more than six plays

a. Each student's doctor reads no technical journals at all
b. No student's doctor reads even one technical journal

a. Each counselor told John at least three stories
b. No counselor told John fewer than three stories

a. Each counselor told either John or Bill at least 3 stories
b. No counselor told both John and Bill fewer than 3 stories

Remark In the last two pairs the middle NPs are "duals", i.e. of the semantic form: F and ¬F¬. Note that proper noun denotations (c-homs) are self dual (B&C).

B. Logical

We compare the GPT with the Linear Prefix Theorem of Keisler & Walkoe (JSL, vol 38, N.1, 1973 pp. 79 - 85). The LPT, somewhat simplified for comparative purposes here is given below.

Linear Prefix Theorem (K&W)

Drawing quantifiers just from {$\forall, \exists$}, Ss of the form in (A) and (B) below are not logically equivalent (indeed, not even finitely equivalent) when the quantifier prefixes are distinct (ϕ is quantifier free and has no constants or function symbols, and P is an n-place predicate symbol):

A. $Q_1x_1 \ldots Q_nx_n \; Px_1 \ldots x_n$ B. $Q_1'x_1 \ldots Q_n'x_n \; \phi$

We may think of the information in the LPT as coming in two parts: Part 1 is the weakened claim of non-equivalence obtained when ϕ in B. above is replaced by $Px_1 \ldots x_n$. This claim just says, in essence, that distinct quantifier prefixes determine distinct n-ary functions, and that much follows from the GPT presented here, observing that the functions induced by $\forall$ and $\exists$ are not trivial and neither is the complement or post complement of the other. Obviously the GPT generalizes, massively, this much of the LPT.

Part 2 of the LPT then says that not only do distinct quantifier prefixes over {$\forall, \exists$} determine distinct functions, but we cannot compensate for this difference by suitably chosing Φ. This part of the LPT is, obviously, not generalized in the GPT. Observe however that this part of the LPT is not preserved under even very modest additions to the quantifiers we may draw from -- additions we would consider trivial in other contexts.

Example 1: If we simply add the negation (<u>not all</u>, <u>not any</u>) of either of the two quantifiers we have <u>then Part 2 of LPT</u> fails: e.g.

$$(\text{not } \forall x)\ Px \equiv (\exists x)(\text{not } Px) \quad \text{and}$$

$$(\text{not } \exists x)\ Px \equiv (\forall x)(\text{not } Px)$$

Example 2: Treat individual constants as quantifiers, viz:

$$M \models (c,x)\Phi\ [\alpha] \quad \text{iff} \quad M \models \Phi\ [\alpha(x/M(c))]$$

Then we have $Qx_1\ cx_2\ Px_1,x_2 \equiv cx_1\ Qx_2\ Px_2,x_1$

<u>Remark</u> Example 2 above is a special case of a basic property of individuals (c-homs) which may be given in the special case of binary relations as follows (writing R^* for the converse of R):

<u>Thm</u> For H a complete homomorphism from R_1 --> 2, F any unary generalized quantifier,

$$(F,H)(R) = (H,F)(R^*)$$

♥