Further Beyond the Frege Boundary

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avant propos This paper is basically Keenan (1992) augmented by some new types of properly polyadic quantification in natural language drawn from Moltmann (1992), Nam (1991) and Srivastav (1990). In addition I would draw the reader's attention to recent mathematical studies of polyadic quantification: Ben-Shalom (1992), Spaan (1992) and Westerstahl (1992). The first and third of these extend and generalize (in some cases considerably) the techniques and results in Keenan (1992). Finally I would like to acknowledge the stimulating and constructive discussions of the earlier paper with many scholars, notably Dorit Ben-Shalom, Jaap van der Does, Hans Kamp, Uwe Monnich, Arnim von Stechow, Mats Rooth, and Ede Zimmermann. And I repeat here the acknowledgment in the earlier paper to Jim Lambek, Ed Stabler and two anonymous referees for *Linguistics and Philosophy* (the latter responsible for substantial improvements in the proofs – see footnote 10).

abstract In sentences like *Every teacher laughed* we think of *every teacher* as a *unary* (= *type <1>*) quantifier – it expresses a property of *one* place predicate denotations. In variable binding terms, unary quantifiers bind one variable. Two applications of unary quantifiers, as in the interpretation of *No student likes every teacher*, determine a *binary* (= *type <2>*) quantifier; they express properties of *two* place predicate denotations. In variable binding terms they bind two variables. We call a binary quantifier *Fregean* (or *reducible*) if it can in principle be expressed by the iterated application of unary quantifiers.

In this paper we present two mathematical properties which distinguish non-Fregean quantifiers from Fregean ones. Our results extend those of van Benthem (1989) and Keenan (1987a). We use them to show that English presents a large variety of non-Fregean quantifiers. Some are new here, others are familiar (though the proofs that they are non-Fregean are not).

The main point of our empirical work is to inform us regarding the types of quantification natural language presents – in particular (van Benthem, 1989) that it goes beyond the usual (Fregean) analysis which treats it as mere iterated application of unary quantifiers.

Secondarily, our results challenge linguistic approaches to "Logical Form" which constrain variable binding operators to "locally" bind just one occurrence of a variable, e.g. the Bijection Principle (BP) of Koopman and Sportiche (1983). The BP (correctly) blocks analyses like *For which x, x's mother kissed x?* for *Who did his mother kiss?* since *For which x* would locally bind two occurrences of *x*. But some of our irreducible binary quantifiers are naturally represented by operators which do locally bind two variables.

This paper is organized as follows: **1** provides an explicit formulation of our questions of concern. **2** classifies the English constructions which we show to be non-Fregean. **3** presents the mathematical properties which test for non-Fregean quantification and applies these tests to the constructions in **2**. Proofs of the mathematical properties are given in the Appendix.

1. *the problem*

Given a universe E of objects (always assumed to have at least two elements) we think of *one place predicates* (P_1s), like *laughed* and *criticized every teacher* in (1), as denoting subsets of E, here called *properties* or *unary relations* (over E).

(1) a. Every teacher laughed
    b. No student criticized every teacher

Similarly *two place predicates* (P_2s) like *criticize* denote binary relations (over E); that is, subsets of E×E. And in general P_ns (n-place predicates) denote n-ary relations = subsets of E^n. The set $P(E^n)$ of n-ary relations is noted $\mathbf{R}_n$. (Note that $\mathbf{R}_0 \approx \{0,1\}$ of truth values).

As NPs like *every teacher* combine with P_1s to form P_0s they are often taken to denote generalized quantifiers – functions from properties ($\mathbf{R}_1$) to truth values ($\mathbf{R}_0$). So interpreting common nouns like *teacher* as properties, *every teacher* denotes that function EVERY TEACHER which sends a property Q to True iff TEACHER $\subseteq$ Q. (NO STUDENT)(Q) = True iff STUDENT $\cap$ Q = $\emptyset$, etc.

But Ss like (1b) suggest that this approach should be generalized, as there *every teacher* combines with a P_2 *criticized,* to form a P_1 *criticized every teacher*. It is natural then to treat the functions denoted by such NPs as having binary (as well as unary) relations in their domain. Applied to binary relations they yield unary relations as values. E.g. EVERY TEACHER takes CRITICIZE to a *set* of objects, namely the set of objects α which bear CRITICIZE to each teacher.

Note that the value of EVERY TEACHER at the CRITICIZE relation is determined by the values it takes at properties: To decide whether an object α is in the property CRITICIZE EVERY TEACHER we look at the set of objects that α bears CRITICIZE to – this is a subset of E – and check that EVERY TEACHER holds of that set.

Further, in *Some dean introduced every teacher to Bill* it seems that *every teacher* has combined with a P_3, *introduce*, to form a P_2 – one that denotes the set of pairs $<\alpha,\beta>$ such that α introduced every teacher to β.

Generalizing, NPs like *every teacher* syntactically combine with P_{n+1}s to form P_ns; semantically they map $\mathbf{R}_{n+1}$ to $\mathbf{R}_n$. Such functions are said to be of *type <1>*. They are defined below, though the full generality of the definitions will not be used in this paper, as we consider only binary and unary relations.

Def-1 F $\in$ **type <1>**, the set of *type <1> functions* (over E) iff

i. the domain of F = $\bigcup_{n\geq 0} \mathbf{R}_{n+1}$, the range of F $\subseteq \bigcup_{n\geq 0} \mathbf{R}_n$, and

ii. for all n $\geq$ 0, all R $\in \mathbf{R}_{n+1}$,

$$F(R) = \{<a_1,\ldots,a_n> | F\{b| <a_1,\ldots,a_n,b> \in R\} = 1\}$$

The semantic interpretation of (1b) may now be given by:

(2) No student criticized every teacher

NO(STUDENT) CRITICIZE EVERY(TEACHER)

(EVERY TEACHER)(CRITICIZE)

(NO STUDENT)((EVERY TEACHER)(CRITICIZE))

The reader may verify[1] using **Def-1** that the last line of (2) represents the correct truth conditions – just those associated with the more familiar variable binding structure in (3), where *student* and *teacher* serve to restrict the range of the variables.

(3) (no student)$_x$(every teacher)$_y$(x criticize y)

Observe that it makes sense to compose type <1> functions. Thus the last line in (2) equals

(4) [(NO STUDENT)∘(EVERY TEACHER)](CRITICIZE)

where [(NO STUDENT)∘(EVERY TEACHER)] maps binary relations to truth values and is thus a function of type <2>[2]. We may now define:

Def-2 For F a function from binary relations (**R**$_2$) to truth values (**R**$_0$), F is *(type <1>) reducible* (= *Fregean*) iff there are type <1> functions f,g such that $F = f \circ g$.

Warning 'reducible' should not be confused here with 'first order'. Some first order quantifiers pick out unreducible functions in some models (e.g. (16)), others do not. And some non-first order ones are reducible, and some are not[3]).

Our concern in this paper is to determine which expressible type <2> functions in English are reducible. We illustrate each type of function with an example, using the following

notation: for R a binary relation over E and $\alpha \in E$, αR is the set of objects α bears R to. Viz., $\alpha R =_{df} \{\beta | \langle\alpha,\beta\rangle \in R\}$.

ex 1 We have seen that interpreting *student* and *teacher* as subsets STUDENT, TEACHER of E, the composition of (NO STUDENT) with (EVERY TEACHER) is a type <2> function which, by construction, is reducible. But if we are directly given a function F from **R**$_2$ to **R**$_0$ it is often not obvious whether it is reducible or not. E.g. given subsets P,Q of E, define $F_{P,Q}$ of type <2> by setting

$$F_{P,Q}(R) = 1 \text{ iff for all } \alpha \in P,\ Q - \alpha R \neq \varnothing$$

So $F_{P,Q}$ is true of R iff for each P there is a Q which that P does not bear R to. Viz., no P bears R to each Q, which is just to say that $F_{P,Q}$ = [NO(P)∘EVERY(Q)], and is thus reducible.

ex 2 Given properties P,Q with at least two elements, the function $F_{P,Q}$ defined below is not reducible (Keenan 1987a):

$$F_{P,Q}(R) = 1 \text{ iff for all distinct } a,b \in P,\ aR \cap Q \neq bR \cap Q$$

Such functions are expressible by the joint action of the subject and object NPs in Ss like (5):

(5) a. Different people like different things
b. Different pupils answered different questions (on the exam)

Thus (5a) seems to us true (on its weakest[4] reading) iff there are at least two people and for all distinct people x,y the things that x likes are not exactly the same as those y likes.

Now, while Keenan (1987a) shows that even the quantification in the weak reading of (5a) is unreducible, the proofs given there have an ad hoc character. We would like generally applicable tests for unreducibility, and we would like to use them to exhibit types of English constructions which take us outside the class expressible by iterated unary quantifiers. It is to these questions that we now turn.

2. non-Fregean quantifiers in English

We first provide a linguistically oriented classification of the types of English expression which we show to be unreducible in **3**. We limit ourselves to Ss of the form NP_1 V NP_2, where V is a transitive verb and the NP_i are Noun Phrases, though ones which either don't occur at all as subjects of intransitive verbs or, if they do, have a different interpretation in transitive contexts than in intransitive ones.

2.1 Neither NP_1 nor NP_2 are semantically interpreted independently of the other

2.1.1 *different-different* First here are the cases cited in (5). Compare (6a) (= 5b) with (6b) and (6c).

(6) a. Different pupils answered different questions (on the exam)
b. (?) Different pupils answered question 6
c. (?) John answered different questions (on the exam)

The logical force of *different* in (6b,c) is to guarantee the existence of at least two pupils/questions (The Ss are slightly marginal). Pragmatically these Ss might be used to emphasize the existence of pupils/questions of different sorts.

But the logical force of the iterated use of *different* in (6a) is to guarantee (minimally) a one-to-one match between pupils and sets of questions (as well as the existence of at least two pupils). Thus in (6a) we do not interpret *different pupils* and *different questions* independently, as we do in (6b) and (6c).

Syntactically and semantically more complex examples of this sort are illustrated in:

(7) a. Different students answered different numbers of questions
b. Different men at the party were wearing different colored neckties
c. People who come from different countries have different tastes in food

The repetition of *different,* forcing a binary quantifier interpretation, is reminiscent of the resumptive quantifiers discussed by van Benthem (1983) and May (1985) in connection with Ss like *No man loves no woman*, which for many speakers can mean no (man,woman) pair is in the LOVE relation. Liu (1991) points out that such examples are rare. *Some man loves three women* does not entail *More than two men love more than two women* even though it guarantees that LOVE contains more than two (man,woman) pairs. And typically, declarative Ss of the form *(d man) love (d woman)* do not have the resumptive reading (D MAN$\times$WOMAN)(LOVE), where the underlying universe is now taken as E$\times$E.

But there are a few atypical cases like *no*, and possibly *exactly one*. A resumptive reading of *exactly one man loves exactly one woman* actually seems to me unavailable, but Irene Heim (pc) points out that such a reading is natural in Ss like *John has published exactly one article in exactly one journal.*

Moreover there are at least two further types of resumptive structures: *resumptive interrogatives* and *resumptive anaphors.*

2.1.2 *resumptive interrogatives* Questions like (8a,b) are familiar, and naturally determine answers which specify (dog,cat) pairs that stand in the CHASE relation.

(8) a. Which dog chased which cat?
b. Which dogs chased which cats?

If (8a) for example were just iterated application of a unary interrogative operator it should be answerable with a single NP, say *Fido*, the result having the interrogative meaning *Fido chased which cat?* But *Fido* is not a sensible answer to (8a), though the pair (*Fido,Puss*) is. So (8a) asks for pairs and is naturally represented in variable binding terms by an operator which locally binds two occurrences of (distinct[5]) variables.

We shall not treat directly of interrogative quantifiers in this paper, but Srivastiv (1990) notes that the type of double binding in (8a,b) above is similar to that expressed by corelative structures in Hindi in (8c,d):

(8) c. jis laRkii-ne dekhaa jis laRke-ko usne usko cahaa
WH girl$_i$-erg saw WH boy$_j$-acc she$_i$ him$_j$ liked

d. jin laRkiyone jin laRkoko dekha, unhone unko cahaa
WH girls$_i$-erg WH boys$_j$-acc saw, they$_i$ them$_j$ liked

(8c) is true according to Srivastav iff there is a unique girl-boy pair in the the SEE relation and that pair is also in the LIKE relation. (8d) is true iff SEE restricted to GIRLS on its first argument and BOYS on its second is bijective and included in LIKE. Thus the *jis-jis* (*jin-jin*) operator semantically maps binary relations to quantifiers of type <2> – maps from binary relations to truth values. The quantifiers we use later to represent the double binding in Bach-Peters sentences are ones of the same logical type as the Hindi corelatives presented here.

2.1.3 *resumptive anaphors* To my knowledge the *else-else* anaphora in (9a) has not been previously noticed.

(9) a. John criticized Bill and noone else criticized anyone else
= Only <John,Bill> stand in the CRITICIZE relation

b. John criticized Bill and noone else criticized Bill
= Only John criticized Bill

c. John criticized Bill and he didn't criticize anyone else
= John criticized only Bill

Observe from (9b,c) that if *noone else* and *anyone else* were independently interpreted in (9a) the result should mean *Only John criticized only Bill*, which allows that people other than John criticized people other than Bill. But that is not what (9a) means. Thus the effect of the *else-else* structure is to directly restrict the pairs in the CRITICIZE relation, not to limit each argument independently.

Moltmann (1992) observes that the polyadicity forced by *else-else* above extends to universally quantified NPs (9d) and the more complicated gapping constructions available in both English and German, (9e).

(9) d. John didn't praise Mary but everyone else praised everyone else

e. Jeder Mann hat mit jeder Frau getanzt ausser Hans mit Maria
every man has with every woman danced except Hans with Mary
'Every man danced with every woman except John with Mary'

As well Nam (1991) notes that simple gapping constructions (9f) as well as the more complex one (9g) from Korean induce properly polyadic quantifiers:

(9) f. Every boy read a novel and every girl a play

g. Dan-i sakwa-lul, kuliko Sue-ka pay-lul Bill-hantey cwuessta
Dan-nom apple-acc and Sue-nom pear-acc Bill-dat gave
"Dan an apple and Sue a pear to Bill gave"

In fact Nam shows that over a finite universe all maps from binary to unary relations are expressible in terms of gapping constructions provided we allow boolean compounds freely (e.g. *Arther saw Frank and Bill Harry but not Bob Sam*).

2.1.4 *Bach-Peters sentences,* (10a), instantiate a different sort of case – one in which the interpretation of each NP depends on that of the other. The *it* in (10a) seems bound to NP_2 as it is in (10b). And the *him* in NP_2 is bound to NP_1 as it is in (10c).

(10) a. Each pilot who shot at it hit exactly one Mig that was chasing him
b. Each pilot who shot at it hit the Mig John was flying
c. Each pilot hit the Mig that was chasing him
d. Each pilot who shot at a Mig that was chasing him hit exactly one Mig that was chasing him that he shot at

One suspects a kind of circularity of reference here; certainly speakers frequently have difficulty interpreting such Ss unless *it* is thought of as referring to some previously identified object[6]. However Higginbotham and May (1981) provide a non-circular semantics for such Ss – one that is paraphrased syntactically in (10d) and seems to conform to our pretheoretical judgments of truth conditions to the extent that they are reliable. Note that, in effect, the offensive *it* in NP_1 in (10a) is replaced by an NP *a Mig that was chasing him* built from NP_2 in such a way that the resulting NP, NP_1 in (10d), is of type <1> with no unresolved anaphors. But the interpretation of NP_1 in (10a) still does depend on that of NP_2 there, since the NP we build depends on NP_2 (Changing *Mig* to *pig* changes how NP_1 is interpreted).

2.2 Branching and Scope Independence

A type <2> function f∘g builds in a scope dependency[7] of g on f (for appropriate f and g – see Nam 1991 and Zimmermann 1987). Branching interpretations are precisely ones where two (or more) quantified NPs are interpreted scope independently. Hence a branching interpretation for potentially scope dependent NPs is not in general reducible. But does English present Ss whose truth conditions require a properly branching interpretation?

Liu (1991) reviews this issue, noting that many of the early examples in the literature (Hintikka, Gabbay and Moravcsik) have been disputed on grounds of empirical adequacy (Fauconnier, Guenthner and Hoepelman). But (11a) from Barwise (1979), (11b) from Jackendoff (1972) and the new examples (11c,d) from Liu survive the critiques.

(11) a. Quite a few boys in my class and most girls in your class have all dated each other
b. I told many of the men three of the stories
c. A majority of the students read those two books
d. A majority of the students read two of those books

Jackendoff (1972) claims, rightly I think, that in addition to a reading on which *three of the stories* is scope dependent on *many of the men,* (11b) has a reading in common with *I told three of the stories to many of the men.* Namely, I told each of three given stories to each of a set of many men. And this is just the scope independent = branching reading (Barwise 1979).

(11c) says that there is some majority of the students each member of which read those two books – a scope independent reading (which coincides with a non-collective subject wide scope (SWS) reading). An object wide scope interpretation is also logically possible: for each of those two books there is a majority of the students who read it, possibly different majorities for different of the books. And speakers find a reading of (11d) (in addition to a SWS one) on which there is a fixed set of two of the books, and a majority of the students read them. That is, as in (11c), the majority does not vary with the choice of book, which is just the branching = scope independent interpretation ≠ the SWS reading.

It seems then that English does present Ss with potentially scope dependent NPs interpreted independently – and those interpretations determine unreducible type <2> functions.

2.3 NP_1 is type <1>; NP_2 is referentially dependent – either on NP_1 or on the transitive verb, or both.

2.3.1 predicate anaphors (PAs)

How we interpret PAs like NP_2 = *more students than Bill* in (12a) depends on the interpretation of the predicate *know*, since NP_2 is

(12) a. John knows more students than Bill (does)
 b. John read every book that Bill did

interpreted as MORE STUDENTS THAN BILL KNOWS. Changing *knows* to *likes* changes the interpretation of NP_2 to MORE STUDENTS THAN BILL LIKES. Semantically, PAs map binary relations to properties satisfying the **Predicate Accusative Anaphor Condition (PAAC)**):

(13) For all binary relations R, all $\alpha,\beta \in E$, if $\alpha R = \beta R$ then $\alpha \in F(R)$ iff $\beta \in F(R)$

That MORE STUDENTS THAN BILL in (12a) satisfies the **PAAC** guarantees that if the individuals John knows are the same as those that Sam knows, then *John knows more students than Bill* iff *Sam knows more students than Bill* (which is clearly the case). Up to isomorphism PAs denote functions from binary relations into **type <1>**.

In variable binding terms PAs lead to logical forms which bind variables of type *transitive verb*. E.g. (12a) can be represented by (14), in which *Know* locally binds 'V' twice.

(14) $(\text{John})_x(\text{Know})_V([\text{more students than Bill V}]_y)(x \text{ V } y)$

Note that PAs like NP_2 in (12) do not grammatically occur as subjects and are not interpretable as type <1> functions at all:

(15) *More students than John does know Sam
 (= More students than John knows know Sam)

2.3.2 NP_2 referentially dependent on NP_1

2.3.2.1 *individual NP anaphors – simplex and complex*

(16) a.b. Each student criticized himself / everyone but himself
 c.d. Each student criticized his teacher / both himself and his teacher

The NP_2s in (16) are naturally interpreted (Keenan 1987b) as functions from binary relations to unary ones. E.g.

(17) (EVERY MAN BUT HIMSELF)(R) = $\{\alpha \in MAN | \alpha R = MAN - \{\alpha\}\}$

Such NP anaphors satisfy the following invariance condition (the **NP Accusative Anaphor Condition (NPAAC)** of Keenan 1987b):

(18) For all $\alpha \in E$, all binary relations R,S if $\alpha R = \alpha S$ then $\alpha \in F(R)$ iff $\alpha \in F(S)$

So e.g. if John criticized just the individuals who he (John) admires, then *John criticized everyone but himself* must have the same truth value as *John admires everyone but himself*.

One shows that the functions satisfying the NPAAC are, up to isomorphism, just the functions from E into **type <1>**. E.g. to interpret (16b) we must consider for each $\alpha \in$ STUDENT, whether α has the property (EVERYONE BUT α)(CRITICIZE) – so the functions we apply to CRITICIZE vary with the choice of entity α.

Representing individual level NP anaphors as variable binding operators we are again led to locally bind two occurrences of the same variable, as in the natural representation (19) for (16b):

(19) (each student)$_x$(everyone but x)$_y$[x criticized y]

Further, predicate anaphors and NP anaphors interact:

(20) a. Each pupil has read more books than his father
b. (each pupil)$_x$Read$_V$(more books than x's father V)$_y$(xVy)

Both predicate anaphors and individual level NP anaphors are naturally treated as functions from $\mathbf{R_2}$ to $\mathbf{R_1}$ (but not also as ones from $\mathbf{R_1}$ to $\mathbf{R_0}$ as per **Def-1**; see Keenan, 1987b). The remaining sorts of referentially dependent NPs we consider (italicized in **2.3.2.2 -2.3.2.6**) seem to lack this property (which does not of course prevent us from asking whether the type <2> functions $\lambda V(NP_1\ V\ NP_2)$ they induce are reducible or not). Tyhurst (1989, 1990), who considers several of these cases and exhibits invariance conditions they satisfy, would treat them as taking binary relations to higher order unary ones – ones true or false (minimally) of sets of individuals[8].

2.3.2.2 *reciprocal anaphors*

(21) The candidates criticized *each other and each other's wives*

2.3.2.3 *cumulative quantification* (Scha, 1984)

(22) a. The two detectives interviewed *a total of forty witnesses*
b. Three producers read *more than fifty plays between them*

2.3.2.4 *comparative dependent determiners*

(23) A certain number of professors interviewed *a much larger number of scholarship applicants*

Dependencies of this sort have not to my knowledge been studied. They arise naturally when the NPs occur in independent Ss, as in *A few students came to the party early, but* ***many more*** *stayed late*.

2.3.2.5 *'same' comparison*

(24) a. Every student answered *the same questions* on the exam
b. They answered *approximately the same number of questions*
c. John and Bill are wearing *the same color shirt*

2.3.2.6 *'different' comparison*

(25) a. The two students answered *different questions* on the exam
b. John and Bill support *rival political parties*
c. Rosa and Zelda date *men who dislike each other*
[On the reading: Rosa dates a man who dislikes a man Zelda dates, and vice versa]

This completes the expression types we will show to be non-Fregean.

3. Formal Tests for Reducibility

We are concerned to to find general tests which decide when a type <2> quantifier reduces to a composition of type <1> ones.

In fact one quite useful test is given in van Benthem (1989). Here we provide two further tests. They are more difficult to apply than van Benthem's and they yield less information than his, but they apply to the full range of type <2> functions over arbitrary E, whereas van Benthem's test is limited to "logical" = "permutation invariant" functions over finite universes. (A type <n> F is *permutation invariant* (PI) iff for all n-ary relations R, F(R) = F(πR), where π is any permutation of E and $\pi R =_{df} \{\langle\pi a_1,\ldots\pi a_n\rangle | \langle a_1,\ldots a_n\rangle \in R\}$). In more detail,

(27) (van Benthem) For E finite and F PI, F is a boolean compound of reducible functions iff for all binary relations R,S F(R) = F(S) whenever $|\alpha R| = |\alpha S|$, for all $\alpha \in E$

Using (27) we show easily that SYM, which holds of a binary relation iff it is symmetric, is not only not reducible, it is not a boolean compound of reducible functions. For proof, set R = $\{\langle a,a\rangle\}$ and S = $\{\langle a,b\rangle\}$. Then for all $\alpha \in E$, $|\alpha R| = |\alpha S|$ but R is symmetric and S is not. Similarly one shows that common logical properties of binary relations, such as being transitive, reflexive, an equivalence relation, etc. are not expressible as boolean compounds of compositions of PI functions.[9]

Van Benthem's test then is informative and of easy application to logical type <2> quantifiers (over finite universes).

But the expressions we considered in **2** commonly denote non-PI functions. E.g. EVERY MAN is non-PI if $\exists\alpha \in$ MAN and a $\beta \notin$ MAN. (Consider a permutation π which interchanges α and β and fixes all other elements of E. Then EVERY(MAN)(MAN) = 1 but EVERY(MAN)(πMAN) = 0, since MAN $\not\subseteq$ π(MAN)).

Here then we provide two tests, Reducibility Equivalence (**RE**) and Reducibility Characterization (**RC**), which apply to non-PI functions. **RE** appears cumbersome but in practice is very useful. **RC** by contrast is fully general. It provides necessary and sufficient conditions for any type <2> to be reducible. But its very generality makes its application more difficult. Van Benthem's test by contrast, is more useful (and more informative) for the cases it applies to (but it applies to fewer cases).

THEOREM Reducibility Equivalence (**RE**)

For F,G reducible functions of type <2>, F = G iff for all subsets P,Q of E, $F(P\times Q) = G(P\times Q)$

Viz, reducible functions are identical iff they take the same values on the cross product relations.

We use **RE** to show a function F unreducible by finding a reducible function G which differs from F but which takes the same values as F on the product relations. In practice finding such a G is often less difficult than it might appear. Still some cases require the full power of **RC** below. Note first:

(28) A type <1> h is called *positive* iff $h(\emptyset) = 0$. So SOME MAN is positive, NO MAN is not. We use **1** for that type <1> function sending all properties to 1; **0** sends all to 0.

THEOREM Reducibility Characterization (**RC**)

A function F from binary relations to $\{0,1\}$ is reducible iff there is a positive type <1> function h such that for all binary relations R,S $F(R) = F(S)$ if $h(R) = h(S)$. (In such a case we say that h *refines* F).

Compare RC with van Benthem's test which gives a fixed condition – $|\alpha R| = |\alpha S|$, all α – forcing F to identify the relations R and S. **RC** by contrast says find a type <1> h such that for all α, $h(\alpha R) = h(\alpha S)$, viz., $h(R) = h(S)$. It is this quantification over **type <1>** which makes our general test harder to apply.

We now use **RE** and **RC** to show that the type <2> functions determined by the expressions discussed in **2** are unreducible.

As an exercise in the use of **RE** we prove unreducible all F which imply that their argument R is reflexive ($\forall \alpha \in E$, $\langle\alpha,\alpha\rangle \in R$), such as those F True of R iff R is reflexive, or iff R is a partial order of E, or an equivalence relation, or a well-order, etc. Restricted to product relations each of these F is true only of $E\times E$, since if an α fails to be in either the domain or the range of R then clearly $\langle\alpha,\alpha\rangle$ is not in R, so R is not reflexive. Thus each of these F coincides with the reducible EVERY(E)$\circ$EVERY(E) on the product relations. But they all differ from that function at $ID_E = \{\langle x,x\rangle | x \in E\}$, so by **RE** none of them are reducible.

Consider now the *different-different* expressions in **2.1**. A typical example is:

(29) Different pupils answered different questions (on the exam)

Chose E with at least three pupils p_1,p_2,p_3 and two questions q_1, q_2. Let F be the type <2> function induced by NP_1 and NP_2 in (29). We show that F has the same values on the product relations as the reducible $\mathbf{0}\circ\mathbf{0}$, which is false of all relations. Let $P\times Q$ be a product relation. If there are two students not in the domain of $P\times Q$ then, trivially, they bear $P\times Q$ to the same questions (the empty set in both cases), so $F(P\times Q) = 0$. If there are two students in $Dom(P\times Q)$ then the questions they bear $P\times Q$ to are again the same: $Q \cap$ QUESTION. So again $F(P\times Q) = 0$. As this covers all the cases we infer that $F(R) = \mathbf{0}\circ\mathbf{0}(R)$ for R a product. But $F \neq \mathbf{0}\circ\mathbf{0}$ as F is true of $R = \{\langle p_1,q_1\rangle, \langle p_2,q_2\rangle\}$. Thus by **RE** F is not reducible. □

Minor variants of this argument show that the '*different-comparison* cases in **2.3.2** (e.g. *Every student answered a different question on the exam*) are also unreducible. Moreover the following observation enables us to systematically extend (un)reducibility claims slightly:

PROPOSITION 1: If F of type <2> is unreducible then so is its complement –F, its postcomplement F–, and its dual –F–, where these notions are defined as follows for all functions F of type <n>, $n \geq 1$

a. $(-F)(R) = -(F(R))$ b. $(F-)(R) = F(-R)$ c. $-F- = -(F-)$

Here R is a set of n-tuples, $-R = E^n - R$, and $-(F(R))$ is just the complement of the truth value F(R). For proof, show by direct calculation that if $F = (f \circ g)$ then $-F = ((-f) \circ g)$ and $F- = (f \circ (g-))$. So $-(F-) = (-f) \circ (g-)$. □

COROLLARY: the set of reducible functions of type <2> is closed under complements, post -complements and duals.

Observe for example that basically (29) is false iff *No two or more students answered exactly the same questions (on the exam)*. Thus the type <2> function expressed in this '*same comparison*' is the complement of that in (29) and thus unreducible by PROP-1.

Other of the examples with '*same comparison*' are more delicate. Interpreting (30a) to imply that John and Bill each answered just two questions on the exam (my best judgment in the matter) we see on a moment's reflection that (30a) and (30b) must have the same truth value when ANSWER is a cross product.

(30) a. John and Bill answered the same two questions (on the exam)
b. Both John and Bill answered exactly two questions (...)

But obviously (30a) and (30b) may have different truth values at non-product relations since (30b) does not even entail that John and Bill answered any questions in common. Thus the function induced by NP_1 and NP_2 in (30a) is the same on product relations as (JOHN AND BILL)∘(EXACTLY TWO QUESTIONS) but is not the same everywhere, and so by RE is unreducible.

Similar examples apply to basic cases of *cumulative quantification*. Thus if READ is a cross product we compute that (31a) and (31b) must have the same truth value.

(31) a. The editors read a total of 46 manuscripts between them
b. Each of the editors read exactly 46 manuscripts

But these Ss may obviously have different truth values, hence the function induced by $\langle NP_1, NP_2 \rangle$ in (31a) behaves the same as the reducible one of (31b) on product relations but differs elsewhere and so is unreducible by **RE**.

Slightly different sorts of examples show reflexive, reciprocal and resumptive anaphora to be unreducible. The core case first:

(32) a. Every student criticized himself
b. Every student criticized every student
= (EVERY STUDENT)∘(EVERY STUDENT)(CRITICIZE)

Clearly if CRITICIZE is a cross product then (32a) and (32b) have the same truth value. So the function F determined by *<every student, himself>* in (32a) takes the same value at products as does the reducible G = (EVERY STUDENT)∘(EVERY STUDENT). But in a model with $\alpha \neq \beta \in$ STUDENT and $R = \{\langle\alpha,\alpha\rangle, \langle\beta,\beta\rangle\}$, $F(R) = 1$ and $G(R) = 0$. Thus by **RE**, in any model with two or more students the type <2> function of (32a) is unreducible.

With minor modifications this argument shows that the type <2> function in *Each of the ten students criticized himself* is unreducible. More interesting: boolean compounds of reflexive anaphors with ordinary type <1> NPs are in general unreducible. (33a) for example takes the same value as (33b) if CRITICIZE is a cross product.

(33) a. Each pupil criticized both himself and the teacher
b. Each pupil criticized every pupil and the teacher

Where SELF is that function from $\mathbf{R}_2$ to $\mathbf{R}_1$ given by SELF(R) $=_{df}$ $\{\alpha \in E| <\alpha,\alpha> \in R\}$, we see that the unreducible function of (32a) is (EVERY STUDENT)$\circ$SELF, where |STUDENT| $\geq$ 2. SELF itself is not type <1> – Keenan, 1987b. It can be extended to take $\mathbf{R}_{n+2}$ to $\mathbf{R}_{n+1}$ but it will not take properties to truth values.

Observe that, occasionally, the composition of a type <1> function with SELF is reducible:
(34) (JOHN$\circ$JOHN) = (JOHN$\circ$SELF)

But by an argument analogous to that for (32), (35a) is unreducible, taking the same value at product relations as the reducible function in (35b) but having different values elsewhere.

(35) a. Each of John and Bill praised himself
(= one reading of: Both John and Bill praised themselves)
b. Each of John and Bill praised both John and Bill

Defining meet ($\wedge$) and join ($\vee$) of type <n> functions pointwise (to interpret conjunctions and disjunctions of type <n> expressions):

(36) $(F \wedge G)(R) = F(R) \wedge G(R)$, and $(F \vee G)(R) = F(R) \vee G(R)$

we observe that the unreducible function in (35a), (JOHN AND BILL)$\circ$SELF = (JOHN$\circ$SELF) $\wedge$ (BILL$\circ$SELF), a meet of two reducible functions. Thus we have shown:

PROPOSITION 2: The type <2> reducible functions are not closed under meets. Nor, given closure under complements (PROP 1), are they closed under joins.

Finally here we note that "exception phrase anaphora" discussed in **2** is, unsurprisingly, unreducible. E.g. in models with two or more students the functions in (37a) and (37b) both coincide with $\mathbf{0}\circ\mathbf{0}$ on the product relations, but may take non-zero values elsewhere and are thus unreducible.

(37) a. Each student criticized everyone but himself
b. Each student criticized noone but himself/only himself

Both *reciprocal anaphors* and *else-anaphora* are shown to be unreducible by arguments similar to those above. E.g. in models where John $\neq$ Bill, (38a,b) take the same values on the product relations but different values on certain other relations. As the function of (38b) is reducible **RE** says that that of (38a) is not.

(38) a. John and Bill criticized each other
b. Both John and Bill criticized everyone who is either John or Bill

Similarly in models where the individuals mentioned in (39) are all distinct we see that (39a) and (39a) must have the same truth value if CRITICIZE is a cross product, but they may have different values elsewhere. Hence the function of (39a) is not reducible since that of (39b) obviously is.

(39) a. John criticized Bill and Sam criticized Harry, but noone else criticized anyone else
b.. Both John and Sam criticized both Bill and Harry

More challenging: show unreducible the functions in (40a,b):

(40) a. John criticized Bill and noone else criticized anyone else
b. Mary criticized each student and Sam in turn criticized her, but noone else criticized anyone else

Using **RC** and **RE** we may prove a general theorem that covers these and many related cases. Observe that the effect of these Ss is to stipulate that certain objects stand in a certain relation, and then the *else-else* form is used to say that no other pairs besides those mentioned stand in the relation. The result is that the type <2> function expressed is an *atom* – it holds of just one relation. But few such functions are reducible. We prove (Appendix):

THEOREM: *Reducible Atoms* (**RA**) For all binary relations R, a reducible F holds just of R iff for some $Q \subseteq E$, $R = E \times Q$

As the Ss in (40) can be true even though CRITICIZE is not of the form E×Q, **RA** shows that the type <2> functions they express are not reducible.

As a final application of **RE** we show that typical *predicate anaphors* are not reducible. Clearly MORE STUDENTS THAN MARY in

(41) John knows more students than Mary (does)

sends a relation R to $\{\alpha|\ |\alpha R \cap \text{STUDENT}| > |\text{mary}R \cap \text{STUDENT}|\}$, the set of objects α who bear R to more students than Mary does. More generally, for $j,m \in E$, set

(42) $F_{j,m}(R) = 1$ iff $|jR \cap \text{STUDENT}| > |mR \cap \text{STUDENT}|$

To see that $F_{j,m}$ is not reducible, suppose that R is a product P×Q. If $R = \varnothing$ then $F_{j,m}(R)$ is clearly 0. If $R \neq \varnothing$ then $jR \cap \text{STUDENT} = Q \cap \text{STUDENT} = mR \cap \text{STUDENT}$ if both j and m are in P. If neither are in P then $jR \cap \text{STUDENT} = \varnothing = mR \cap \text{STUDENT}$ – in both cases F(R) = 0. The only case where F(R) = 1 is when $j \in P$, $P \times Q \neq \varnothing$, and $m \notin P$. That is, for R a product relation,

F(R) = ((JOHN AND NOT MARY)∘(SOMETHING))(R)

the righthand side being true iff John bears R to something and Mary does not. Thus F takes the same values as some reducible function at the product relations, but may obviously take different values on other relations – e.g. one in which John bears R to two students and Mary bears R to just one. Thus by RE this predicate anaphor is non-Fregean.

For the remaining sorts of expressions for which unreducibility is claimed in **2** we find it convenient to use the **RC** theorem. **RC** claims, recall, that F is reducible iff there is a positive type <1> h which refines F (that is, F(R) = F(S) if h(R) = h(S)).

We consider first the *comparative dependent determiners* such as that in (43):

(43) A certain number of professors interviewed a much larger number of scholarship applicants

A fair approximation to the type <2> function determined by the NPs in (42a) is given in (44), where for P = PROFESSOR and Q = SCHOLARSHIP APPLICANT, $F_{P,Q}$ is true of R iff the number of professors who bear R to a scholarship applicant is less than the number of applicants that some professor bears R to.)

(44) $F_{P,Q}(R) = 1$ iff $|\text{Dom}(R \cap (P \times Q)| < |\text{Ran}(R \cap P \times Q)|$

We now apply the Reducibility Characterization (RC) theorem to show that functions like $F_{P,Q}$ are not reducible. We use (here and later) the following basic but very useful fact:

fact-1 For h positive and $P,Q \subseteq E$,

$$h(P\times Q) = \begin{cases} P & \text{if } h(Q) = 1 \\ \varnothing & \text{if } h(Q) = 0 \end{cases}$$

proof Let $h(Q) = 0$. Then $\forall\alpha$, $\alpha[P\times Q] = \varnothing$ or $\alpha[P\times Q] = Q$, so for all α, $h(\alpha[P\times Q]) = 0$, whence $h(P\times Q) = \varnothing$. If $h(Q) = 1$ and $\alpha \notin P$ then $h(\alpha[P\times Q]) = h(\varnothing) = 0$, so $\alpha \notin h(P\times Q)$; if $\alpha \in P$ then $h(\alpha[P\times Q]) = h(Q) = 1$, so $\alpha \in h(P\times Q)$. So $h(P\times Q) = P$. □

Now, chose E with at least three elements and find subsets P,Q of E such that $|P \cap Q| \geq 3$. We show that $F_{P,Q}$ as in (44) is unreducible. Suppose leading to a contradiction that there is a positive h which refines $F_{P,Q}$. Chose subsets S,T of $P \cap Q$ with $0 < |S| < |T|$ and $h(S) = h(T)$. Such S,T exist since h is two valued and for each n, $0 < n \leq 3 \leq |P \cap Q|$ there non-empty properties of cardinal n.

Now, since $h(S) = h(T)$ then $h(S\times S) = h(S\times T)$. But direct computation shows that $F_{P,Q}(S\times S) = 0$ while $F_{P,Q}(S\times T) = 1$, contradicting that h refines $F_{P,Q}$. So there is no such h, whence by **RC**, $F_{P,Q}$ is unreducible. #

A similar technique shows that even the simplest cases of *branching* quantifiers are not reducible.

Let $A = \{a_1,a_2\}$ be a two element subset of E, and let $B = \{b_1,b_2,b_3\}$ a three element subset of E. Let F be the (1,1) quantifier AT LEAST TWO. So F(A) and F(B) are type <1> functions. The branching type <2> quantifer F(A)*F(B) determined by these type <1> quantifiers is defined by:

(45) $F(A)*F(B)(R) = 1$ iff $\exists A' \subseteq A$, $|A'| \geq 2$, $\exists B' \subseteq B$, $|B'| \geq 2$ such that $A'\times B' \subseteq R$ (that is, every A' bears R to every B')

Now suppose there is a positive type <1> h which refines F. Let B_1 and B_2 be distinct two element subsets of B such that $h(B_1) = h(B_2)$. Such sets exist since there are three two element subsets of B and h is two valued. Set $R = A\times B_1$ and $S = \{<a_1,b> | b \in B_1\} \cup \{<a_2,b> | b \in B_2\}$. We compute directly that $F(A)*F(B)(R) = 1$ and $F(A)*F(B)(S) = 0$. Thus h fails to refine F, a contradiction. So no such h exists, whence F(A)*F(B) is unreducible by **RC**.

The remaining case to consider is the Bach-Peters example with the semantics assigned by Higginbotham and May (1981). Here for F and G of type (1,1) the "absorbed" quantifier F*G they define is of type (2,2). It sends a binary relation R to a type <2> quantifier as per the stipulation below.

(46) $(F*G)(R)(S) = F(DOM(R),\{\alpha | G(\alpha R,\alpha S) = 1\})$

In general K = (EVERY*SOME)(R) is not reducible. One computes:

(47) $K(S) = (EVERY*SOME)(R)(S) = 1$ iff $DOM(R) \subseteq \{\alpha | \alpha R \cap \alpha S \neq \varnothing\}$

Now set $R = \{<a,a>, <b,b>\}$, $S = \{<a,b>, <b,a>\}$, $S' = \{<b,b>\}$, and $S'' = \{<a,a>\}$, $(a \neq b)$. Then $K(R) = 1$ and $K(S) = K(S') = K(S'') = 0$. Suppose that there is a positive h of type <1> which refines K. If $h\{a\} = h\{b\}$ then we compute that $h(R) = h(S)$, whence $K(R) = K(S)$, a contradiction. Thus $h\{a\} \neq h\{b\}$. If $h\{a\} = 1$ and $h\{b\} = 0$ then $h(R) = \{a\} = h(S'')$, whence

K(R) = K(S"), a contradiction. So h{a} = 0 and h{b} = 1. But then h(R) = {b} = h(S'}, implying that K(R) = K(S'), a contradiction. Thus no such h exists, so by **RC** K is unreducible.

This completes the consideration of the type <2> functions considered in **2**. Proofs of the theorems used in our tests for unreducibility are given in the Appendix. Here let us close this section with a query, prompted by Arnim von Stechow. Namely, *To what extent will a compositional semantics for, say, English be forced to have recourse to the functions we have shown unreducible*?

A convincing answer to this query would require an adequate grammar for English, compositionally interpreted. And while no one has ever seen such a beast, it does seem plausible that not all the constructions we have adduced as determining polyadic quantifiers must in fact be so interpreted. E.g. consider the gapping cases. While the second "conjunct" in (48a) below naturally determines a binary quantifier, the interpretation of (48b) requires no such function. So if we design our grammar + interpreting function in such a way that we semantically interpret (48b) and syntactically derive (48a) assigning it the same interpretation as the expression we derived it from, then the interpretation of (48a) does not in fact require the use of unreducible binary quantifiers, though it does involve some complexity in the syntax.

(48) a. John loves Mary and Bill Sue
b. John loves Mary and Bill loves Sue

On the other hand, for the large majority of the expressions we have considered as determining unreducible polyadic quantifiers we do not find simple syntactic paraphrases not involving such quantification. Our feeling is then that very little of the polyadic quantification we have discussed can be syntactically paraphrased away.

Appendix: *Proofs of RC, RE, and RA*

lemma 1 If F of type <1> is reducible then there exist type <1> functions f,g with g positive such that $F = f\circ g$.

proof Recall that a type <1> h is positive iff $h(\varnothing) = 0$. Recall also that h– sends each property p to h(–p), and –h sends each p to –(h(p)). Now observe that for all f,g of type <1>,

$$(f{-}\circ{-}g)(R) = (f-)((-g)(R)) = (f-)(-(g(R)) = f(-(-(g(R))) = f(g(R)) = (f\circ g)(R)$$

Thus if F of type <2> is reducible then for some f,h of type <1>, $F = f\circ h = f{-}\circ{-}h$ and since one of h, –h is positive there is a positive g such that $F = f\circ g$. ⋆

lemma 2 If F of type <2> is reducible and constant on the product relations then F is constant

proof Let $F = f\circ g$, with g positive (lemma 1) and suppose that for some $\alpha \in \{0,1\}$, $F(P\times Q) = \alpha$, all $P,Q \subseteq E$. We show that $F(R) = \alpha$, all binary relations R.

Let R be arbitrary. Set $R_g = R - \bigcup_{g(aR)=0}\{a\}\times aR$

Then $g(R) = g(R_g)$, since if $g(aR) = 0$ then $a \notin Dom(R_g)$ so $aR_g = \varnothing$ and $g(\varnothing) = 0$ since g is positive. If $g(aR) = 1$ then $a \in Dom(R)$, since g is positive, and $aR = aR_g$. Thus for all $a \in E$, $g(aR) = g(aR_g)$ so $g(R) = g(R_g)$.

Thus $F(R) = (f\circ g)(R) = f(g(R)) = f(g(R_g)) = (f\circ g)(R_g) = F(R_g)$.

Now, if $R_g = \emptyset$ then $F(R) = F(\emptyset) = F(\emptyset \times \emptyset) = \alpha$ and we are done. If $R_g \neq \emptyset$ let $b \in Dom(R_g)$, so $g(bR_g) = 1$. Set $R' = DomR_g \times bR_g$. Clearly $g(R') = Dom(R_g) = g(R)$. So $F(R) = (f \circ g)(R) = f(g(R)) = f(g(R')) = (f \circ g)(R') = F(R') = \alpha$, since R' is a cross product relation. □

Theorem *Reducibility Equivalence* **(RE)**

Let F,G be type <2> reducible functions. Then,

$$F = G \text{ iff } F(P \times Q) = G(P \times Q) \text{ all } P,Q \subseteq E$$

case 1 one of F,G is constant. Say $F(R) = \alpha \in \{0,1\}$, all R. But then $F(R) = \alpha$ for all product relations R, so $G(R) = \alpha$ for all product relations, so by lemma 2 $G(R) = \alpha$, all R, so $G = F$.

case 2 Neither F nor G are constant. Let $F = f \circ f'$ and $G = g \circ g'$, with f' and g' positive. We show that $f' = g'$ and $f = g$, whence $F = G$.

f' = g'. Suppose otherwise. wlg let $f'(Q)) = 1$ and $g'(Q) = 0$. So $Q \neq \emptyset$ since f' is positive. Now for P any subset of E, $f(P) = (f \circ f')(P \times Q) = F(P \times Q) = G(P \times Q) = (g \circ g')(P \times Q) = g(\emptyset)$ since g' is positive and false of Q. But this just says that f is constant, since P was arbitrary. ∴ for all R,S $F(R) = (f \circ f')(R) = f(f'(R)) = f(f'(S))$, since f is constant, $= (f \circ f')(S) = F(S)$, and thus F is constant, contradicting the initial assumption. Thus there can be no Q such that $f'(Q) \neq g'(Q)$, so $f' = g'$.

f = g Let Q non-empty such that $f'(Q) = 1$. There is such a Q since otherwise f' is constant, whence F is constant. And since $f' = g'$ we have $g'(Q) = 1$. Then, for all $P \subseteq E$,

$f(P) = f(f'(P \times Q)) = (f \circ f')(P \times Q) = F(P \times Q) = G(P \times Q) = (g \circ g')(P \times Q) = g(P)$, so $f = g$. Thus $F = G$. □

Theorem *Reducibility Characterization* **(RC)**

For F of type <2>, F is reducible iff there is a positive type <1> h which refines F (i.e. for all R,S $F(R) = F(S)$ if $h(R) = h(S)$).

a. ⇒ Let $F = f \circ g$ with g positive. Choose g to be the h mentioned in the theorem. Clearly

if $g(R) = g(S)$ then $F(R) = (f \circ g)(R) = f(g(R)) = f(g(S)) = (f \circ g)(S) = F(S)$.

b. ⇐ Suppose there is a positive h as in the theorem[10]. Define f_h of type <1> by: $f_h(P) = 1$ iff $\exists S F(S) = 1 \ \& \ h(S) = P$.

Then $f_h(h(R)) = 1$ iff $\exists S F(S) = 1 \ \& \ h(S) = h(R)$
iff $F(R) = 1$

whence $F = f_h \circ h$, so F is reducible. □

Theorem *Reducible Atoms* **(RA)**

a. For all binary relations R we define F_R of type <2> by setting $F_R(S) = 1$ iff $S = R$. And we prove:

b. For all R, F_R is reducible iff $\exists Q' \subseteq E$, $R = E \times Q'$

$\Leftarrow$ Compute that $F_R = ALL(E)\circ[ALL(Q') \wedge NO(-Q')]$, and is thus reducible.

$\Rightarrow$ Let F_R be reducible.

Observe first that by **RE**, F_R must hold of some product relation $P\times Q$, otherwise it takes the same values as **0**$\circ$**0** on the products and thus is false of all relations, contradicting that it holds of R. And since F_R holds just of R we have that $R = P\times Q$. We must show that for some Q', $R = E\times Q'$.

If either P or Q is $\emptyset$ then $R = \emptyset = E\times\emptyset$ and we are done. Thus we may assume that neither P nor Q is empty, and so $R = P\times Q \neq \emptyset$. By **RC** $\exists$ positive h of type <1> which refines F_R. $h(Q) = 1$, since otherwise $h(P\times Q) = \emptyset = h(\emptyset\times\emptyset)$ and thus, since h refines F_R, $F_R(P\times Q) = F_R(\emptyset\times\emptyset)$, contradicting that F_R holds of just one relation. Further, for all $W \neq Q$, $h(W) = 0$. Otherwise for some such W $h(W) = 1$ and so $h(P\times Q) = h(P\times W) = P$, whence $F_R(P\times Q) = F_R(P\times W)$, contradicting again that F_R holds of just one relation.

Now, assume leading to a contradiction, that $P \neq E$; say $\alpha \in E - P$. Since E has at least two elements, find $\beta \notin Q$ if Q happens to be a unit set (if Q isn't a unit set it doesn't matter whether $\beta \in Q$ or not). So $Q \neq \{\beta\}$. Set $T = P\times Q \cup \{<\alpha,\beta>\}$. Thus $P\times Q \neq T$. Since $\alpha T = \{\beta\}$ we have that $h(\alpha T) = 0$, so $h(T) = h(P\times Q)$ and thus $F_R(T) = F_R(P\times Q) = 1$, contradicting that F_R holds of just one relation. Thus $P = E$ and $R = E\times Q$, proving the theorem. #

Footnotes

1. Specifically, for α an element of E and R a binary relation, write αR for $\{\beta|\alpha R\beta\}$. Then,

(NO STUDENT)((EVERY TEACHER)(CRITICIZE)

= (NO STUDENT)($\{\alpha|$ (EVERY TEACHER)(αCRITICIZE) = 1$\}$)

= 1 iff STUDENT $\cap$ $\{\alpha|$(EVERY TEACHER)(αCRITICIZE) = 1$\}$ = $\emptyset$

= 1 iff no student is an α which bears CRITICIZE to every teacher

2. In general the type <n> functions are those mapping $\cup_{k\geq 0}\mathbf{R}_{k+n}$ into $\cup_{k\geq 0}\mathbf{R}_k$ satisfying for all $n \geq 1$, all $k \geq 0$, all $R \in \mathbf{R}_{k+n}$,

$$F(R) = \{<a_1,...,a_k>|F\{<b_1,...,b_n>|<a_1,...,a_k,b_1,...,b_n> \in R\}$$

Thus our representation in (2) yields the same truth conditions as the variable binding representation in (5), where we think of *student* and *teacher* as restricting the range of the variables.

3. A simple example of a type <1> reducible function which is not first order definable is given as follows, using the usual "global" notion of first order definability:

Let F associate with each universe E a function F_E from $\mathbf{R}_2$, the set of binary relations over E, to $\mathbf{R}_0 = \{0,1\}$. Then F is *first order definable* iff there is a formula ϕ in the first order language L(P) whose only non-logical constant is a two place predicate symbol P such that for all models $M = <E,R>$, $M \models \phi$ iff $F_E(R) = 1$.

Now let F send each E to (JUST FINITELY MANY)(E)$\circ$SOME(E). Then for each E, F_E is reducible. But a standard compactness argument shows that F is not first order definable: Let K be the infinite set of sentences {(For at least n x)($\exists$y)xPy|n $\in \omega$}. Each $\phi \in K$ is first order. Set $K' = K \cup$ {(For just finitely many x)($\exists$y)xPy}. Clearly each finite subset of K' has a model but K' itself has no model. So L(P) enriched with the binary quantifier *For just finitely many x there is a y* is not first order.

4. A stronger reading is one in which we require $aR \cap Q$ to be disjoint from $bR \cap Q$. Van Benthem (1989) suggests an even stronger (clearly non-first order) reading in which different Ps bear R to different Qs iff $R \cap P \times Q$ embeds an injection with the same domain.

5. The variables must be distinct because they have distinct ranges (dogs vs cats). The binary quantifiers under discussion here seem to me reasonable counterexamples to the Bijection Principle (BP) as given. But we can probably avoid these counterexamples and still account for the phenomena Koopman and Sportiche consider by reformulating the BP to read: "An operator may not locally bind two occurrences of the same variable".

6. I occasionally test such Ss in undergraduate Linguistics courses and normally find that noone is able to naturally assign them an interpretation with *it* bound to NP_2.

7. Note that in our notation $f \circ g$, g affects the second argument of the binary relation $f \circ g$ applies to. Noting this explicitly by $f_1 \circ g_2$ the object wide scope reading is given by $g_2 \circ f_1$, where for R binary, $f_1(R) =df \{b|f\{a|\langle a,b\rangle \in R\} = 1\}$. This approach is taken explicitly in Keenan (1987b) and developed significantly in Nam (1991). Now technically if $f_1 \circ g_2 \neq g_2 \circ f_1$ the latter function is not reducible on our definition (see Nam 1991 for proof). But this function is clearly the composition of two type <1> functions, so we should technically extend our definition of type <1> reducible to cover this case. As it is the only case we consider that crucially uses the additional notation we merely note this fact here rather than carry the added notation throughout the paper.

8. Tyhurst's treatment is natural in many respects, but it does pass one big buck – namely, how do we suddenly start intepreting ordinary NPs like *John but no other students* as sets of properties of *sets* of individuals?

9. This result contrasts with Nam's (1991) proof that, for E finite, all elements of $[\mathbf{R}_2 \rightarrow \mathbf{R}_0]$ are expressible as boolean compounds of the reducible functions in that set. But van Benthem's result shows that the PI subset of $[\mathbf{R}_2 \rightarrow \mathbf{R}_0]$ is not generable by boolean compounds of compositions of PI unary functions.

10. This part of the proof is due to one of the referees. It replaces my more constructive but incomparably less elegant one.

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