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LOGICAL OBJECTS

INTRODUCTION

This paper began, and begins, as a reflection on the following claim by Putnam (*1981*, page 217): "*Theorem*: Let $\mathcal{L}$ be a language with predicates F_1, $F_2, \ldots, F_k$ (not necessarily monadic). Let $\mathcal{I}$ be an interpretation, in the sense of an assignment of an intension to every predicate of $\mathcal{L}$. Then if $\mathcal{I}$ is non-trivial in the sense that at least one predicate has an extension which is neither empty nor universal in at least one possible world, there exists a second interpretation $\mathcal{J}$ which disagrees with $\mathcal{I}$, but which makes the same sentences true in every possible world as $\mathcal{I}$ does. ... If at least one predicate, say, F_u, has an extension R_{uj} which is neither empty nor all of $\mathcal{U}_j$, select a permutation P_j of $\mathcal{U}_j$ such that $P_j(R_{uj}) \neq R_{uj}$. ..."

Lewis (*1984*) accepts some form of Putnam's claim, and Lakoff (*1987*, chapter 15: "Putnam's Theorem ... The Inconsistency of Model Theoretic Semantics") uses it to argue against the applicability of model theoretic semantics to the study of natural language.

But we show, in section 1, that Putnam's claim, as stated, is false. We reformulate the claim, proving that *permutation invariance* (PIness) rather than non-triviality is what binds reference (extensions of predicates) to truth. In section 2 we argue, contra Etchemendy (*1990*), that invariance under automorphisms (AIness, a conceptual generalization of PIness) characterizes the denotations of "logical" expressions, including but not limited to classical logical constants. In section 3 we defend our main thesis: that the logical objects of a given type in a given ontology are the AI ones. Our analysis derives without circularity (Theorem 22) that being "logical" on our thesis is itself a logical property. Section 4 concludes with an open problem.

Previous work, originating with Klein (*1872*) and including *Weyl 1949, Mostowski 1957, Silva 1945, van Fraassen 1990, Keenan 1991, Keenan and Stabler 1991*; *Keenan and Stabler 1995* identifies the *structural* properties of objects in a given domain with those invariant under the structure preserving permutations of that domain. That the *logical* properties of objects may be identified with those invariant under all permutations was, to my knowledge, first proposed in a lecture by Tarski in 1966, edited and published as *Tarski 1986* by J. Corcoran. Tarski there explicitly builds on Klein's Erlanger Program (*1872*; see *Klein 1892* and *van Fraassen 1990*, page 266) in

C. Anthony Anderson and M. Zelëny (eds.), Logic, Meaning and Computation, 149-180.

which geometries of various sorts are distinguished according as the objects of study are invariant under one or another class T of transformations (= permutations). E.g., "Euclidean objects" are ones invariant under translations, rotations, etc. Then the weaker the invariance conditions the transformations must satisfy the larger the equivalence classes of objects. And Tarski makes the natural extrapolation: the most general objects, the "logical" ones, are those invariant under all transformations.[1] It is also this association of generality with PIness under the heading "symmetry" in the physical science literature that lies behind van Fraassen's use of the role of symmetry in characterizing scientific laws.

The idea that "PIness" = "logicality" is developed most extensively in *van Benthem 1989a*. Plotkin *1980,* Keenan and Stavi *1986,* van Benthem *1991,* Marshall and Chuaqui *1991,* and Sher *1991* also support this idea. We differ only slightly from these in emphasizing the type and structure dependency of "logical" objects. This enables us to explicate the sense in which, e.g., set membership both is, and is not, a logical relation. We draw on natural language studies to exhibit a variety of novel logical objects in high types (*van Benthem 1983, 1991*; *Westerståhl 1985*; *Keenan and Moss 1985*; *Keenan and Stavi 1986*).

Notation. $[A \to B]$ is the set of functions from A into B. If $f \in [A \to B]$ and $\mathrm{K} \subseteq A$, $f(\mathrm{K})$ is $\{f(x) \mid x \in \mathrm{K}\}$; for $s \in A^n$, $f(s)$ is $\langle f(s_1), \ldots, f(s_n)\rangle$. The power set of A is denoted $\mathcal{P}_A$ or $\mathcal{P}(A)$. Languages $\mathcal{L}$ are assumed first order relational (without zero place relation symbols) unless indicated otherwise. *Models* for such $\mathcal{L}$s are pairs $\mathrm{M} = (\mathrm{E_M}, \mathcal{I})$, $\mathrm{E_M}$ a non-empty set, the *universe* of M (usually denoted E), and $\mathcal{I}$ a function mapping the relation (function) symbols of $\mathcal{L}$ to relations (functions) over E. As usual, $\mathcal{I}$ lifts recursively to an *interpretation* $\mathcal{I}_\mathrm{M}$ of $\mathcal{L}$ relative to assignments of values to the variables. For ϕ a sentence write $\mathrm{M} \models \phi$ for $\mathcal{I}_\mathrm{M}(\phi) = 1$, the latter meaning that $\mathcal{I}_\mathrm{M}(\phi)(g) = 1$, for all assignments g. $\mathsf{Th}[\mathrm{M}]$, the theory of M, $=_{\mathrm{df}} \{\phi \in \mathsf{Sent}(\mathcal{L}) \mid \mathrm{M} \models \phi\}$. Models M, M' are *elementarily equivalent*, $\mathrm{M} \cong \mathrm{M}'$, if $\mathsf{Th}[\mathrm{M}] = \mathsf{Th}[\mathrm{M}']$. For $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ and α a bijection with domain E, $\alpha\mathrm{M} =_{\mathrm{df}} (\alpha\mathrm{E}, \alpha \circ \mathcal{I})$ is a model. For M, M' models, M' is isomorphic ($\simeq$) to M iff $\mathrm{M}' = \alpha\mathrm{M}$, some bijection $\alpha : \mathrm{E_M} \to \mathrm{E_{M'}}$. And of course

(1) $\mathrm{M} \simeq \mathrm{M}' \Rightarrow \mathrm{M} \cong \mathrm{M}'$ (so $\mathrm{M} \cong \alpha\mathrm{M}$, all bijections α with domain $\mathrm{E_M}$).

[1]The notion of invariance developed here differs in two respects from that in *Tarski 1986*: (1) Tarski does not treat Truth and Falsity, or truth functions (AND, OR, ...) built on them, as objects to be preserved by permutations. He just considers a set theoretic hierarchy built up from the universe and notes that none of its elements are PI, only two subsets are, only four binary relations are. At higher levels relations like *subset* and *disjoint* will be PI. (2) Tarski's paper is informal. He does not explicitly define a type hierarchy and identify the PI objects as those fixed by all permutations. He even remarks that while the logical properties of classes are the numerical ones, the idea "is quite difficult to formulate in a precise way" (page 151).

1. BINDING REFERENCE AND TRUTH

To evaluate Putnam's claim let $\mathcal{L}_P$ be of signature $\{P\}$, P a P_1 (one place predicate symbol). Then, in a possible world w with universe $\mathcal{U}_w$, if $\mathcal{I}$ interprets P as non-empty and non-universal, so that $\varnothing \subset \mathcal{I}(P) \subset \mathcal{U}_w$, let π be a permutation of $\mathcal{U}_w$ interchanging some $x \in \mathcal{I}(P)$ with some $y \notin \mathcal{I}(P)$ and fixing everything else. As per (1) then, $(\mathcal{U}_w, \mathcal{I}) \cong (\mathcal{U}_w, \pi \circ \mathcal{I})$, but $\mathcal{I} \neq \pi \circ \mathcal{I}$ since $x \in \mathcal{I}(P)$ and $x \notin (\pi \circ \mathcal{I})(P)$, in conformity with Putnam's claim.

But Putnam's claim fails when P is a P_n, $n \geq 2$. Let $\mathcal{U}_w$ be the doubleton $\{a, b\}$ and let $\mathcal{I}(P) = \{\langle a, b\rangle, \langle b, a\rangle\}$. $\mathcal{I}(P)$ is not $\varnothing$ nor universal $= \{a, b\} \times \{a, b\}$, but all models $(\{a, b\}, \mathcal{J})$ with $\mathcal{J} \neq \mathcal{I}$ disagree on some sentence of $\mathcal{L}_P$.

Proof. (a), (b) and (c) below are false in $\mathrm{M} = (\mathcal{U}_w, \mathcal{I})$:

(a) $\exists x(xPx)$ (b) $\forall x \forall y \neg(xPy)$ (c) $\exists x \exists y(xPy \,\&\, \neg yPx)$

Let $(\{a, b\}, \mathcal{J})$ be a model of $\mathcal{L}$ with $\mathcal{J}(P) \neq \mathcal{I}(P)$. If $\langle c, c\rangle \in \mathcal{J}(P)$ for some $c \in \mathcal{U}_w$ then (a) above is true in $\mathcal{J}$. So we may assume that for all $c \in \mathrm{E}$, $\langle c, c\rangle \notin \mathcal{J}(P)$. If no pair $\langle c, d\rangle$ of distinct elements of $\mathcal{U}_w$ is in $\mathcal{J}(P)$ then $\mathcal{J}(P)$ is empty and (b) above is true. So at least one of $\langle a, b\rangle$, $\langle b, a\rangle$ is in $\mathcal{J}(P)$. If both are then $\mathcal{J} = \mathcal{I}$ contrary to assumption. So just one is, and in each case (c) is true. This exhausts the cases. ⊣

Lemma 2 reformulates Putnam's claim in terms of *permutation invariance* (PIness) not non-triviality (the two notions coinciding on P_1s). And Theorem 4 tells us that PIness is equivalent to covariation of truth and reference (holding the universe constant). Theorems 7 and 12 partially generalize Theorem 4 to show that for sufficiently rich $\mathcal{L}$s, PIness guarantees a kind of expressive completeness: such $\mathcal{L}$s describe their models up to isomorphism. First,

DEFINITION 1.

(i) An n-ary relation R over a universe E is *permutation invariant* (PI) iff $\pi R = R$, for all permutations π of E. (Recall: $\pi(R) = \{\pi(d) \mid d \in R\}$). For $R \in [\mathrm{E}^m \rightarrow \mathrm{E}]$ this just says that $R(\pi d) = \pi(R(d))$, all $d \in \mathrm{E}^m$.

(ii) A *model* (E,I) is *PI* iff for all $P \in \mathrm{Dom}(\mathcal{I})$, $\mathcal{I}(P)$ is PI.

LEMMA 1. *For $R \subseteq \mathrm{E}^n$, R is not* PI iff $\exists s \in \mathrm{E}^n \, \exists \pi \in \mathrm{PERM(E)}$, $s \in R$ *and* $\pi s \notin R$.

Proof.

$\Leftarrow$ If $s \in R$ then $\pi s \in \pi R \neq R$ since $\pi s \notin R$; so R is not PI.

$\Rightarrow$ Let R such that for $\delta \in \mathrm{PERM(E)}$, $\delta R \neq R$.

Case 1: $\exists t \in R - \delta R$. Then the righthand side above holds choosing $s = t$ and $\pi = \delta^{-1}$.

Case 2: $\exists t \in \delta R - R$. Then $\delta^{-1}t \in \delta^{-1}\delta R = R$ and $t \notin R$ so the righthand side above holds choosing $\pi = \delta$ and $s = \delta^{-1}t$. ⊣

LEMMA 2. *Let* $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ *a model for* $\mathcal{L}$ *and* P *a relation symbol of* $\mathcal{L}$ *with* $\mathcal{I}(P)$ *not* PI. *Then* $\mathcal{L}$ *has a model* $\mathrm{M}' = (\mathrm{E}, \mathcal{J}) \cong \mathrm{M}$ *such that* $\mathcal{J}(P) \neq \mathcal{I}(P)$.

Proof. Given $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ with $\mathcal{I}(P)$ not PI. By Lemma 2 there is a tuple s and a permutation π of E such that $s \in \mathcal{I}(P)$ and $\pi(s) \notin \mathcal{I}(P)$. Then $\pi\mathrm{M} = (\mathrm{E}, \pi \circ \mathcal{I}) \simeq \mathrm{M}$ so $\mathsf{Th}[\pi\mathrm{M}] = \mathsf{Th}[\mathrm{M}]$. But $\mathcal{I} \neq \pi \circ \mathcal{I}$ since $\pi(s) \notin \mathcal{I}(P)$ but $\pi(s) \in (\pi \circ \mathcal{I})(P) = \pi(\mathcal{I}(P))$ since $s \in \mathcal{I}(P)$. ⊣

COROLLARY 3. *Every non-*PI *model has an elementarily equivalent model with the same universe which disagrees with it on some predicate symbol.*

THEOREM 4 **Stability of Reference.** *Let* $\mathcal{L}$ *have equality. Then for all models* M *for* $\mathcal{L}$,

$$\mathrm{M} \textit{ is } \mathrm{PI} \Leftrightarrow \forall \mathrm{M}' \textit{ with } \mathrm{E}_{\mathrm{M}'} = \mathrm{E}_{\mathrm{M}},\ \mathrm{M}' \cong \mathrm{M} \leftrightarrow \mathrm{M}' = \mathrm{M}.$$

Proof.

⇒ Let $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ be PI. Let $\mathrm{M}' = (\mathrm{E}, \mathcal{J})$. The ← direction of the consequent is trivial. Assume $\mathrm{M}' \neq \mathrm{M}$, so $\mathcal{J} \neq \mathcal{I}$. We show $\mathrm{M}' \ncong \mathrm{M}$. Let P be of arity n with $\mathcal{J}(P) \neq \mathcal{I}(P)$. Let $t \in (\mathcal{I}(P) - \mathcal{J}(P)) \cup (\mathcal{J}(P) - \mathcal{I}(P))$, let x be an n-tuple $x_1, \ldots, x_n$ of distinct variables and let $\mathsf{Dist}(x)$ be the formula $\text{AND}\{(x_i = x_j) \mid t_i = t_j, 1 \leq i \neq j \leq n\}$ & $\text{AND}\{(x_i \neq x_j) \mid t_i \neq t_j, 1 \leq i \neq j \leq n\}$, where for K a finite set of formulas, $\text{AND}(\mathrm{K}) = \forall x(x = x)$ if $\mathrm{K} = \varnothing$, and otherwise $\text{AND}(\mathrm{K})$ is the conjunction of the formulas in K. Consider $\phi = \forall x(\mathsf{Dist}(x) \rightarrow Px)$. If $t \in \mathcal{I}(P) - \mathcal{J}(P)$, then $\mathrm{M} \models \phi$ and $\mathrm{M}' \not\models \phi$; if $t \in \mathcal{J}(P) - \mathcal{I}(P)$, then $\mathrm{M}' \models \phi$ and $\mathrm{M} \not\models \phi$.

⇐ Follows immediately from Lemma 2. ⊣

Theorem 4 tells us that it is precisely PIness that guarantees that changing extensions of predicates changes the truth of sentences. Does Theorem 4 generalize, replacing identity of universes with sameness of cardinality and equality of models with isomorphism? In fact only half the generalization obtains. We first establish the useful:

LEMMA 5. *A model* M *is* PI iff *for all* $\pi \in \text{PERM}(\mathrm{E}), \pi\mathrm{M} = \mathrm{M}$.

Proof.

⇒ Let M be PI and $\pi \in \text{PERM}(\mathrm{E})$. Then for all $P \in \mathsf{Dom}(\mathcal{I})$, $\mathcal{I}(P) = \pi(\mathcal{I}(P)) = (\pi \circ \mathcal{I})(P)$. Since P was arbitrary $\mathcal{I} = \pi \circ \mathcal{I}$, so $\pi\mathrm{M} = (\pi\mathrm{E}, \pi \circ \mathcal{I}) = (\mathrm{E}, \mathcal{I}) = \mathrm{M}$.

⇐ Let $P \in \mathsf{Dom}(\mathcal{I})$. For $\pi \in \text{PERM}(\mathrm{E})$, $\pi(\mathcal{I}(P)) = (\pi \circ \mathcal{I})(P) = \mathcal{I}(P)$, so $\mathcal{I}(P)$ is PI. ⊣

LEMMA 6. *Let* M *be a model for* $\mathcal{L}$. *Then* [a] $\Rightarrow$ [b] *and if* $\mathcal{L}$ *has equality,* [b] $\Rightarrow$ [a].

[a] *For all models* M', $M \cong M'$ $\Rightarrow$ *for all bijections* $\pi : E_M \to E_{M'}$, $M' = \pi M$,

[b] M *is* PI.

Proof.

[a] $\Rightarrow$ [b]. Assume [a] and let $M = (E, \mathcal{I})$, let $\delta \in \text{PERM}(E)$. Show $\delta M = M$, whence M is PI by Lemma 5. Since $M \simeq \delta M$, then $M \cong \delta M$, so for all bijections $\pi : E \to E_{\delta M} = \delta E = E$, $\delta M = \pi M$. Choosing $\pi = \text{id}_E$ yields $\delta M = M$.

[b] $\Rightarrow$ [a]. Let $\mathcal{L}$ have $=$, let $M = (E, \mathcal{I})$ be PI and let $M' = (E', \mathcal{I}')$ with $M \cong M'$. If $|E| \neq |E'|$, then [a] holds vacuously. So let π be a bijection: $E \to E'$. Show $\pi M = M'$. Assume, leading to a contradiction, that equality fails. Then $\pi \circ \mathcal{I} \neq \mathcal{I}'$, so for some $P \in \text{Dom}(\mathcal{I})$, $\mathcal{I}'(P) \neq (\pi \circ \mathcal{I})(P)$. Now if there is a $t \in (\pi \circ \mathcal{I})(P) - \mathcal{I}'(P)$, then, where $\text{Dist}(x)$ is formed as in Theorem 4, $\phi = \forall x(\text{Dist}(x) \to Px)$ is true in πM and false in M'. But $\pi M \simeq M$ so $\pi M \cong M$ so ϕ is true in M, contradicting $M \cong M'$. And if there is a $t \in \mathcal{I}'(P) - (\pi \circ \mathcal{I})(P)$, then ϕ is false in πM and true in M'. And again since $\text{Th}[\pi M] = \text{Th}[M]$, ϕ is false in M, contradicting that $\text{Th}[M] = \text{Th}[M']$. This exhausts the cases proving the lemma. ⊣

THEOREM 7. *Let* M *be a model of a relational language* $\mathcal{L}$ *with equality. Then*

$$M \textit{ is PI} \Rightarrow \forall M' \textit{ with } |E_{M'}| = |E_M|, M' \cong M \text{ iff } M' \simeq M.$$

Proof. Given M PI and M' with $|E_{M'}| = |E_M|$. Let $\pi : E_{M'} \to E_M$ be a bijection. Assume $M' \cong M$. By Lemma 6, $\pi M = M'$ and, (1), $M \simeq \pi M$. ⊣

FACT 1. The converse to Theorem 7 is false.

Proof. Let $\mathcal{L}$ have signature $\{P\}$, P of arity 2. Let $M = (\{a, b\}, \mathcal{I})$ with $\mathcal{I}(P) = \{\langle a, b\rangle\}$. Then for all M' with $|E_{M'}| = 2$, $\text{Th}[M'] = \text{Th}[M] \Rightarrow M' \simeq M$. In general (*Enderton 1972*, page 96, example 17) finite models which are elementarily equivalent are isomorphic. Here is a demonstration for this case. Note that Th[M] includes:

$$\text{(a) } \neg \exists x Pxx \qquad \text{(b) } \exists xy Pxy \quad \text{and} \quad \text{(c) } \forall x, y(Pxy \to \neg Pyx).$$

Let $M' = (\{\alpha, \beta\}, \mathcal{I}')$ with $\alpha \neq \beta$. Assume $\text{Th}[M'] = \text{Th}[M]$. By (a) neither $\langle \alpha, \alpha \rangle$ nor $\langle \beta, \beta \rangle$ is in $\mathcal{I}'(P)$. By (b) either $\langle \alpha, \beta \rangle$ or $\langle \beta, \alpha \rangle$ is in $\mathcal{I}'(P)$ and, by (c), not both. So $\mathcal{I}'(P) = \{\langle \alpha, \beta \rangle\}$ or $\mathcal{I}'(P) = \{\langle \beta, \alpha \rangle\}$. In the first case the map sending a to α and b to β is an isomorphism; in the second the map sending a to β and b to α is. So $M' \simeq M$. But $\mathcal{I}(P)$ is not PI. The π interchanging a and b fails to fix $\{\langle a, b \rangle\}$. ⊣

FACT 2. In Theorem 7 the condition that $|E_M| = |E_{M'}|$ is necessary.

Proof. Let $\mathcal{L}_=$ be the zero-signature first order $\mathcal{L}$ with equality. For any E, $(E, \mathcal{I})$ is a model of $\mathcal{L}_=$, since $\mathcal{I}(=)$ is required to be $\{\langle a, a\rangle \mid a \in E\}$. Now let M and M′ be models of $\mathcal{L}_=$, where $E_M = \mathbb{N}$, the natural numbers, and $E_{M'} = \mathcal{P}(\mathbb{N})$. By Theorem 9, $\mathsf{Th}[M] = \mathsf{Th}[M']$; but $M \neq M'$ since their universes differ in cardinality. ⊣

LEMMA 8. *Let* M, M′ *be infinite models for* $\mathcal{L}_=$. *Then for all formulas* ϕ *in* $\mathcal{L}_=$, *for all assignments* $g : \text{VAR} \to E_M$ *and all assignments* $g' : \text{VAR} \to E_{M'}$

if $g(x) = g(y)$ iff $g'(x) = g'(y)$, *for all* $x, y \in \text{FREEVAR}(\phi)$,
then $M \models \phi[g]$ iff $M' \models \phi[g']$.

Proof. By recursion on formula complexity (Appendix). ⊣

THEOREM 9. *All infinite models of* $\mathcal{L}_=$ *have the same theory.*

Proof. Let $\phi \in \mathsf{Sent}(\mathcal{L}_=)$, let M, M′ infinite models of $\mathcal{L}_=$. Clearly all $g : \text{VAR} \to E_M$ and $g' : \text{VAR} \to E_{M'}$ identify the same variables free in ϕ. So for all g, g', $M \models \phi[g]$ iff $M' \models \phi[g']$, so $M \models \phi$ iff $M' \models \phi$. Thus $\mathsf{Th}[M] = \mathsf{Th}[M']$. ⊣

N.B.: It is *only* the inability to state the cardinality of the universe which prevents the more general statement that for PI models, sameness of theories and model isomorphism coincide. This latter statement is naturally understood as a kind of expressive completeness requirement: languages whose models satisfy it can describe themselves up to isomorphism (Theorem 12).

DEFINITION 2 **Cardinality Quantifiers.** Let $\mathcal{L}$ be a first order language. We expand $\mathcal{L}$ to $\mathcal{L}(\text{CARD})$ by adding the cardinality quantifiers Q_λ for each cardinal λ, as follows:

syntax: For each cardinal λ, Q_λ is a *quantifier symbol* (as are ∀ and ∃). For all quantifier symbols Q, all variables x, and all formulas ϕ, $Qx\phi$ is a formula.

semantics: $M \models Q_\lambda x\phi[g]$ iff $|\{a \in E_M \mid M \models \phi[g[x/a]]\}| = \lambda$.

In expanding $\mathcal{L}$ to $\mathcal{L}(\text{CARD})$ we add new clauses to the definition of satisfaction but we do not add new symbols to be interpreted. Hence the models $M = (E, \mathcal{I})$ for $\mathcal{L}(\text{CARD})$ are exactly those for $\mathcal{L}$; the definition of PIness for models is unchanged. We need:

LEMMA 10. PI*ness is preserved by model isomorphism.*

Proof. Let $M = (E, \mathcal{I})$ be PI, let $\delta : M \simeq M' = (E', \mathcal{I}')$. Show M' is PI. Let $\pi \in \text{PERM}(E')$. Show $\pi(\mathcal{I}'(P)) = \mathcal{I}'(P)$, for all $P \in \text{Dom}(\mathcal{I}')$. Note that $\mathcal{I}'(P) = (\delta \circ \mathcal{I})(P) = \delta(\mathcal{I}(P))$. And since $\delta^{-1} \circ \pi \circ \delta \in \text{PERM}(E)$ and M is PI, $(\delta^{-1} \circ \pi \circ \delta)(\mathcal{I}(P)) = \mathcal{I}(P)$, so $\delta((\delta^{-1} \circ \pi \circ \delta)(\mathcal{I}(P))) = \delta(\mathcal{I}(P)) = \mathcal{I}'(P)$. But $\delta((\delta^{-1} \circ \pi \circ \delta)(\mathcal{I}(P))) = (\pi \circ \delta)(\mathcal{I}(P)) = \pi(\mathcal{I}'(P))$, whence $\pi(\mathcal{I}'(P)) = \mathcal{I}'(P)$, so M' is PI. ⊣

LEMMA 11. *Let* $\pi : M \simeq M'$. *Show* $\forall \phi \in \mathcal{L}(\text{CARD})$, $\forall g : \text{VAR} \to E_M$,

$$M \models \phi[g] \text{ iff } M' \models \phi[\pi \circ g].$$

Proof. Set $S = \{\phi \in \mathcal{L}(\text{CARD}) \mid \forall g : \text{VAR} \to E_M, M \models \phi[g] \text{ iff } M' \models \phi[\pi \circ g]\}$. See *Enderton 1972* (pages 91–92) for the proof that S includes the atomic formulas and is closed under conjunction, negation and universal quantification. We show that S is closed under cardinal quantification.

Let $\phi \in S$. Show for all g, all cardinals λ that $M \models Q_\lambda x\phi[g]$ iff $M' \models Q_\lambda x\phi[\pi \circ g]$.

Observe that $\forall g : \text{VAR} \to E_M, \forall x \in \text{VAR}, \forall a \in E_M$, $\pi \circ g[x/a] = (\pi \circ g)[x/\pi a]$:

$$\text{(a) } (\pi \circ g[x/a])(y) = \begin{cases} \pi(g[x/a](y)) = \pi(a) & \text{if } y = x, \\ \pi(g(y)) & \text{if } y \neq x, \end{cases}$$

$$\text{(b) } (\pi \circ g)[x/\pi a](y) = \begin{cases} \pi a & \text{if } y = x, \\ (\pi \circ g)(y) = \pi(g(y)) & \text{if } y \neq x. \end{cases}$$

And we infer: $M \models \phi[g[x/a]]$ iff $M' \models \phi[\pi \circ g[x/a]]$ Induction Hypothesis.
iff $M' \models \phi[(\pi \circ g)[x/\pi a]]$ the observation.

Thus the map $\pi \restriction \{a \in E_M \mid M \models \phi[g[x/a]]\} \to \{\pi a \in \pi(E_M) = E_{M'} \mid M' \models \phi[(\pi \circ g)[x/\pi a]]\}$ is onto (and one-to-one, since π is one-to-one). Thus

$$|\{a \in E_M \mid M \models \phi[g[x/a]]\}| = |\{\pi a \in E_{M'} \mid M' \models \phi[(\pi \circ g)[x/\pi a]]\}|.$$

Whence for all cardinals λ, $M \models Q_\lambda x\phi[g]$ iff $M' \models Q_\lambda x\phi[\pi \circ g]$, completing the proof. ⊣

THEOREM 12 **Expressive Completeness.** *For all* $\mathcal{L}$ *with* $=$, *for all models* M *for* $\mathcal{L}(\text{CARD})$,

$$M \textit{ is PI} \Rightarrow \textit{for all models } M' \textit{ for } \mathcal{L}(\text{CARD}), M \cong M' \text{ iff } M \simeq M'.$$

Proof. Given $\mathcal{L}$ with $=$, let M and M′ be models of $\mathcal{L}(\text{CARD})$ with M PI.

⇒ Let $\text{Th}[M] = \text{Th}[M']$. For $\lambda = |E_M|$, $M' \models Q_\lambda(x = x)$ since $M \models Q_\lambda(x = x)$, so $|E_{M'}| = |E_M|$. Let $\pi : E_M \simeq E_{M'}$. By Lemma 11, $M' = \pi M$, so $M' \simeq M$ by Lemma 10.

$\Leftarrow$ For M, M′ isomorphic models of $\mathcal{L}$(CARD) and $\phi \in$ Sent($\mathcal{L}$(CARD)), Lemma 11 implies that M $\models \phi$ iff M′ $\models \phi$, whence Th[M] = Th[M′], completing the proof. ⊣

Theorem 12 entails that elementary equivalence of models of $\mathcal{L}$(CARD) preserves PIness. But we have a stronger result, whose proof incorporates some observations used later:

THEOREM 13. *For $\mathcal{L}$ with =, M and M′ models of $\mathcal{L}$, if M is PI & M $\cong$ M′ then M′ is PI.*

Proof sketch. As noted earlier, the PI n-ary relations form a complete atomic subalgebra of $\mathcal{P}(\mathrm{E}^n)$, denoted PI($\mathrm{E}^n$).

13.1 For each $s \in \mathrm{E}^n$, set $[s] =_{\mathrm{df}} \{t \in \mathrm{E}^n \mid t_i = t_j \text{ iff } s_i = s_j, \text{ for all } 1 \leq i, j \leq n\}$. Then $[s]$ is non-empty ($s \in [s]$) and PI, and no proper non-empty subset of $[s]$ is PI, so $[s]$ is an atom of PI(E^n). Whence for R an n-ary PI relation on E, $R = \bigcup_{s \in R}[s]$.

13.2 For $s \in \mathrm{E}^n$ let $s_\equiv$ be that equivalence relation on $\{1, \ldots, n\}$ given by: $i\, s_\equiv\, j$ iff $s_i = s_j$. Let $F(s)$ be the formula $(\text{AND}\{(x_i = x_j) \mid i\, s_\equiv\, j\}\ \&\ \text{AND}\{(x_i \neq x_j) \mid \neg i\, s_\equiv\, j\})$, where AND(K) is as defined earlier for K a finite set of formulas. Also we write OR(K) for $\exists x(x \neq x)$ if K = $\emptyset$; otherwise OR(K) is the disjunction of the elements of K.

13.3 For M = (E, $\mathcal{I}$) a model of $\mathcal{L}$ with = and P a P_n, one proves that

$$\mathcal{I}_{\mathrm{M}}(P) \text{ is PI iff } \mathrm{M} \models \forall x_1, \ldots, x_n(Px_1, \ldots, x_n \leftrightarrow \text{OR}\{F(s) \mid s \in \mathcal{I}_{\mathrm{M}}(P)\}).$$

Write 'P' is *PI* for the formula to the right of '$\models$' above. It says that 'P' denotes $\emptyset$ or the non-trivial union of the $[s]$ for the tuples s in its extension. That is, it says "'P' is PI".

13.4 Now, let $\mathcal{L}$ have =, let M and M′ be models with M $\cong$ M′ and M PI. Then for each predicate symbol P (including '='), 'P' is *PI* is true in M and thus in M′, so M′ is PI. ⊣

2. LOGICAL OBJECTS AND LOGICAL CONSTANTS

We have seen that PI relations play a special role with respect to the truth of sentences. Here we extend this observation to a characterization of "logical objects", objects we intend to include as denotations of classical logical constants as well as an infinitely enriched class drawn from natural language. So we cannot ad hocly define the logical objects to be the denotations of a listed set of constants. Rather we need a criterion for including new objects among the "logical" ones. So contra Etchemendy (*1990*), we seek a property common to denotations of logical expressions. And we claim that a certain generalization of PIness is that property.

First, just as the "equals" predicate is a standard logical one and for each E, its denotation $\mathsf{ID}_{\mathrm{E}} = \{\langle a, a\rangle \mid a \in \mathrm{E}\}$ is PI, so, when interpreted directly, the denotations of other standard logical expressions will be "logical objects". For example, interpreting the standard quantifiers directly (*Westerståhl 1985*) we think of $\exists x\phi$ as $\exists\lambda x\phi$, where $\lambda x\phi$ is interpreted as a set (or its characteristic function) and $\exists$ is interpreted as a function from sets to truth values, viz., that function mapping a subset A of the universe to *True* iff A is not empty. Similarly $\forall$ maps A to *True* iff A is the universe. (Note the universe dependence of $\forall$.) These functions must count as logical objects, though we must say something about their PIness.

Similarly we want to say that the truth functions denoted by classical *and*, *or*, and *not*, are logical objects, though naive application of PIness here appears to yield a wrong result: the table for AND contains the line $\langle\langle \mathrm{T},\mathrm{F}\rangle,\mathrm{F}\rangle$. But under a permutation of $\{\mathrm{T},\mathrm{F}\}$ which interchanges T and F the resulting table contains $\langle\langle \mathrm{F},\mathrm{T}\rangle,\mathrm{T}\rangle$ which is not a line in the table for *and*, so the denotation of *and* is (obviously) not invariant under all permutations of $\{\mathrm{T},\mathrm{F}\}$.

Now in distinction to the universe of a model, the set of truth values is *structured*. It has two elements so that Ss (sentences) and their negations can be interpreted differently and its boolean structure is given by a $\leq$ relation. One of the two elements is "least" in that it bears the $\leq$ relation to both elements; it is the one assigned to a disjunction of Ss just in case each disjunct is assigned that value. The other element is "greatest" in that it bears the $\leq$ only to itself; it is the value assigned to a conjunction of Ss iff each conjunct is assigned that value. Thus the permutations π of truth values we want are just the boolean *automorphisms*, those that satisfy $x \leq y$ iff $\pi x \leq \pi y$ (which is to say $\pi(\leq) = \leq$, that is, π *fixes* $\leq$). We recall that all two element boolean lattices are isomorphic and that the only automorphism of such a lattice is the identity function.

In general the automorphisms of a structure $(A, R_1, R_2, \ldots)$ are the permutations of the domain A which fix the relations R_i. The automorphisms of a structure (like the universe of a model) with no relations (functions) given are simply the permutations of the domain. Note that being structured, like the truth values, is typical of the primitives we need to model natural language. For example to represent tense marking (present, past, ...) we might include as a primitive a set T of points of time (or intervals), equipped with a precedence (or overlap) relation. To account for intensional operators we might include a set W of possible worlds, equipped with an accessibility relation. Doubtless more is needed. And the permutations we need are just the automorphisms. This guarantees that the properties of models determined by the primitives hold as well of their images under an automorphism, and thus those images are also models for the language. And the notion of PIness we require of objects to be "logical" is *invariance under automorphisms*, AIness.

Before formally defining AIness we consider some of the novel expressions whose denotations we wish to count among the logical objects. These include ones which are not first order definable and ones of high logical type. So we must consider automorphisms of structures other than the primitive ones we build our models on.

First, there are infinitely many expressions of the same category—Determiner (Det) in linguistic parlance—as *some* and *all*. (See *Keenan and Westerståhl 1994* and references cited there for an overview of recent work in this area; note in particular the collections *van Benthem and ter Meulen 1985*, *Gärdenfors 1987* and *van der Does and van Eijck 1996*; *Keenan and Moss 1985* and *Keenan and Stavi 1986* are probably the most extensive empirical studies.)

(1) a. *Not more than fifty* students here major in math.

b. *Most* undergraduates study some Linguistics but *less than two percent* major in it.

c. *All but two* of the students signed up for advanced Ulithian this quarter.

d. *Both* students answered question six correctly but *neither* got an A on the exam.

Many of these Dets are first order: *no, both, neither* and those of the form *more/fewer than n, exactly n, between n and m, all but n, the n*. But many are not: *just finitely many, all but finitely many, infinitely many,* as well as (*Barwise and Cooper 1981*) the properly proportional Dets like *most, more/less than half, exactly ten percent, less than a third* (even if the universe is required to be finite). But all share the logical character of the classical first order quantifiers.

Now, given an *ontology* Ω, namely a set E_Ω of entities and a boolean set 2_Ω of truth values (subscripts usually omitted), we treat P_1s like *major in math, get an A on the exam*, etc. as denoting properties of entities, represented as functions from E into 2. So P_1s, like common nouns (*student, undergraduate*, etc.), denote in $[E \to 2]$. NPs (Noun Phrases) like *some student, John and some student*, etc. combine with P_1s to form Ss (P_0s) and denote in $[[E \to 2] \to 2]$, the set of *generalized quantifiers over* E. So we think of a sentence like *Every student laughed* as denoting the truth value assigned to the denotation of *laugh* by the function denoted by *every student*. And as Dets like *every* combine with nouns to form NPs we can treat them as denoting in $[[E \to 2] \to [[E \to 2] \to 2]]$.

More systematically now let us represent denotation sets relative to a choice of ontology $(E, 2)$ in terms of *types*. The notion of type used in linguistic semantics derives from Church (*1940*) but replaces his "ι, o" notation with Montague's (*1973*) more mnemonic "e, t"—"e" for entity, "t" for truth value. We extend it to include product types:

DEFINITION 3. *type* is the least set containing the *initial types* 'e' and 't' and satisfying (a) and (b):

(a) $\alpha, \beta \in$ *type* $\Rightarrow (\alpha, \beta) \in$ *type* (functional types),

(b) $\alpha_1, \ldots, \alpha_k \in$ *type* $\Rightarrow (\alpha_1 \bullet \ldots \bullet \alpha_k) \in$ *type* (product types).

Type indexes denotation sets relative to an ontology $\Omega = (\mathrm{E}, 2)$. An expression is of type τ if, for each ontology Ω, it is interpreted as an element of $\mathsf{Den}_\Omega\tau$, defined by:

(2) a. $\mathsf{Den}_\Omega e = \mathrm{E}$,
b. $\mathsf{Den}_\Omega t = \{0_2, 1_2\}$,
c. $\mathsf{Den}_\Omega(\alpha, \beta) = [\mathsf{Den}_\Omega(\alpha) \to \mathsf{Den}_\Omega(\beta)]$,
d. $\mathsf{Den}_\Omega(\alpha_1 \bullet \ldots \bullet \alpha_k) = \mathsf{Den}_\Omega\alpha_1 \times \ldots \times \mathsf{Den}_\Omega\alpha_k$.

DEFINITION 4. For each ontology Ω, the Ω *objects* are just those in the union of the $\mathsf{Den}_\Omega\tau$.

Our concern here is to characterize those Ω objects appropriately called *logical*. But we anticipate. First, certain of the $\mathsf{Den}_\Omega\tau$ are always boolean, as defined by:

DEFINITION 5. *bool-type* $=_{\mathrm{df}}$ the least subset of *type* satisfying:

a. $t \in$ *bool-type*,
b. If $\alpha \in$ *type* and $\beta \in$ *bool-type* then $(\alpha, \beta) \in$ *bool-type*, and
c. If $\alpha_1, \ldots, \alpha_k \in$ *bool-type* then $(\alpha_1 \bullet \ldots \bullet \alpha_k)$ *bool-type*.

For τ boolean, $\mathsf{Den}_\Omega\tau$ is a complete atomic boolean lattice, given by:

(3) a. For $x, y \in \mathsf{Den}_\Omega t$, $x \leq y$ iff $x = 0_2$ or $y = 1_2$,
b. For $f, g \in \mathsf{Den}_\Omega(\alpha, \beta)$, $f \leq g$ iff for all $a \in \mathsf{Den}_\Omega(\alpha)$, $f(a) \leq g(a)$,
c. For $f, g \in \mathsf{Den}_\Omega(\alpha_1 \bullet \ldots \bullet \alpha_k)$, $f \leq g$ iff for each $1 \leq i \leq k$, $f_i \leq g_i$.

And we are now in a position to define *automorphism invariance*:

DEFINITION 6. A map π with domain type is an automorphism of an ontology $\Omega = (\mathrm{E}, 2)$ iff

1. For all $\tau \in \{e, t\}$, $\pi(\tau)$ is an automorphism of $\mathsf{Den}_\Omega\tau$,
2. $\pi(\alpha, \beta)$ is that map sending each f in $\mathsf{Den}_\Omega(\alpha, \beta)$ to $\{\langle \pi(\alpha)(a), \pi(\beta)(b) \rangle \mid \langle a, b \rangle \in f\}$, and
3. $\pi(\alpha_1 \bullet \ldots \bullet \alpha_k)$ sends each k-tuple a in $\mathsf{Den}_\Omega(\alpha_1 \bullet \ldots \bullet \alpha_k)$ to $\langle \pi(\alpha_1)(a_1), \ldots, \pi(\alpha_k)(a_k) \rangle$.

FACT 3. For all automorphisms π of Ω, all types τ, $\pi(\tau)$ is a permutation of $\mathsf{Den}_\Omega(\tau)$. If $\tau \in$ bool-type then $\pi(\tau)$ is a boolean automorphism of $\mathsf{Den}_\Omega(\tau)$. Moreover, every permutation of E extends uniquely to an automorphism of $\Omega = (\mathrm{E}, 2)$.

DEFINITION 7.

(i) For all ontologies Ω and types τ, a $d \in \mathrm{Den}_\Omega\tau$ is *automorphism invariant* (AI) iff $\pi(\tau)(d) = d$, all automorphisms π of Ω. Write $\mathrm{AI}_\Omega\tau$ for the AI elements of $\mathrm{Den}_\Omega\tau$.

(ii) For d a closed expression of $\mathcal{L}$, d is *AI* iff for all models M for L, $\mathcal{I}_{\mathrm{M}}(d)$ is AI.[2]

Notation: We usually write $\pi(d)$ for $\pi(\tau)(d)$. And for π some Ω-automorphism we write π^{-1} for that map sending each $\tau \in$ *type* to $(\pi(\tau))^{-1}$, the inverse of the permutation $\pi(\tau)$. $\pi \circ \pi^{-1}$ is understood similarly and is thus the identity automorphism.

THEOREM 14. *Let $\mathcal{L}$ have '$=$' of type $((e \cdot e), t)$, '$\exists$' and '$\forall$' of type $((e,t),t)$,* and *and* or *of type $((t \cdot t), t)$ and* not *of type (t,t), all interpreted as indicated earlier. Then each of these expressions is* AI.

Below we present many theorems of this sort omitting proofs for reasons of space. We illustrate one case here to give the reader a feel for the algebraic character of these proofs. Observe first that for π an automorphism, (1) $a \in A$ iff $\pi(a) \in \pi(A)$, and (2) $(\pi F)(\pi a) = \pi b$ says the same as $\langle \pi a, \pi b\rangle \in \pi F$. Now we show that, given E, $F : \mathcal{P}(\mathrm{E}) \to 2$ is AI, where $F(\mathrm{A}) = 1$ iff $\mathrm{A} \neq \emptyset$. Let π an arbitrary automorphism. We show that $\pi F = F$:

$(\pi F)(A) = 1$	iff $(\pi F)(\pi(\pi^{-1}A)) = 1$	π, π^{-1} are inverses
	iff $\pi(F(\pi^{-1}(A)) = 1$	observation (2)
	iff $F(\pi^{-1}(A)) = 1$	π is the identity on truth values
	iff $\pi^{-1}(A) \neq \emptyset$	def F
	iff $A \neq \emptyset$	observation (1); π^{-1} an automorphism
	iff $F(A) = 1$	def F

Of course the truth functions denoted by *and*, *or*, and *not* are not the only AI ones. All truth functions are AI (*van Benthem 1989a*). Let us define the pure *t-types* to be the closure of $\{t\}$ under the formation of functional and product types. Then,

THEOREM 15. *For all ontologies* $(\mathrm{E}, 2)$, *all* $\tau \in$ *t-type, each* $d \in \mathrm{Den}_{(\mathrm{E},2)}\tau$ *is* AI.

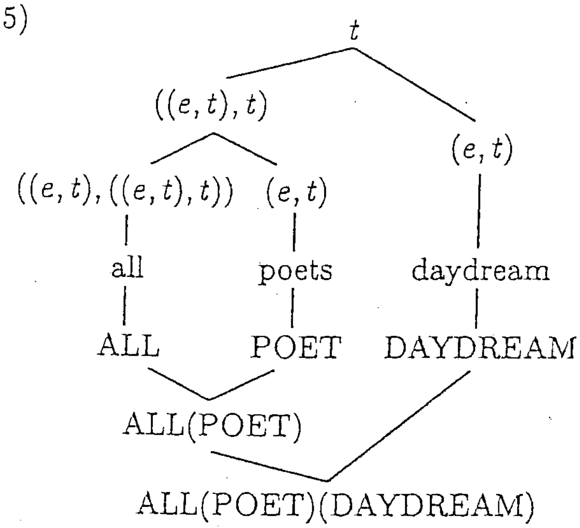
And as well, the Dets discussed above, of type $((e,t),((e,t),t))$, are AI. Here are some sample definitions, assuming an ontology with universe E and A, B arbitrary subsets of E.

[2] We need not require that d have no free variables; but then the definition must mention $\mathcal{I}_{\mathrm{M}}(d)(g)$, for all assignments g.

(4) a. $\text{ALL}(A)(B) = 1$ iff $A - B = \emptyset$.
b. $\text{NO}(A)(B) = 1$ iff $A \cap B = \emptyset$.
c. $\text{SOME}(A)(B) = 1$ iff $A \cap B \neq \emptyset$.
d. $\text{MOST}(A)(B) = 1$ iff $|A \cap B| > |A - B|$.
e. $(\text{ALL BUT TWO})(A)(B) = 1$ iff $|A - B| = 2$.
f. $(\text{EXACTLY TEN})(A)(B) = 1$ iff $|A \cap B| = 10$.
g. $(\text{THE TWO})(A)(B) = 1$ iff $|A| = 2$ and $A - B = \emptyset$.

(4a) says in effect that *All poets daydream* is true iff the poets less the daydreamers is the empty set. (5) below illustrates the type structure and semantic interpretation we assume:

(5)



THEOREM 16. *The* Dets *given in* (4) *are* AI.

Observe that not all expressions which share the syntactic distribution of the Dets above do present a "logical" character. Contrast:

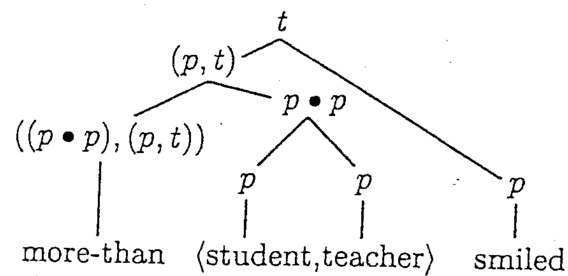
(6) a. *All / John's* cats are black,
b. *Most / More male than female* students read the Times.

In (6a) the interpretation of *John's* depends both on which individual *John* denotes and on what objects he possesses. Many non-AI functions can interpret *John's* (*Keenan and Stavi 1986*). In (6b) *more male than female* admits of non-AI interpretations, as it depends on which objects are male and which female, a contingent matter.

We return now to "logical" expressions in natural language. They share with the Dets in (4) the property of being AI. Consider first the comparative quantifiers in (7) (*Keenan and Moss 1985, Beghelli 1992*). Their type structure is represented in (8), using 'p' ("property") for (e, t):

(7) a. *More/Fewer* students *than* teachers signed the petition,
b. *Exactly as many* students *as* teachers signed the petition,
c. *More than twice* (*n times*) *as many* students *as* teachers signed,
d. *The same number of* students *as* teachers signed the petition.

(8)



(more students than teachers smiled)

The interpretation of these cardinal comparatives is straightforward: MORE As THAN Bs have C iff $|A \cap C| > |B \cap C|$. Quite generally cardinal comparatives are not first order definable: *more-than* defines *most* (over a universe E) by MOST(A)(B) iff (MORE $A \cap B$ THAN $A - B$)(E). That is, most As are Bs iff more As who are Bs than As who are not Bs exist. So if MORE-THAN were first order definable MOST would be, but by Barwise and Cooper (*1981*) it is not.

The examples given so far suggest that logical expressions of high type depend on cardinalities. But this is not in general the case (See *Keenan 1987*, page 92; *van Benthem 1989b*):

(9) a. *Different* people like *different* things,
b. *No two* students read *the same* papers,
c. *Each* student answered *the same* questions (on the exam).

Keenan (*1992*) shows that we do not obtain a correct semantic analysis of these sentences if for example *different things, ..., the same questions* are interpreted as generalized quantifiers. A semantically correct analysis interprets *different-different* (*no two-the same*, etc.) as functions mapping pairs of properties to functions from binary relations into 2. So writing r for $(e,(e,t))$, *different-different* has type $((p \bullet p),(r,t))$. As illustrated in (10a) and defined in (10b) this function is AI, and its values depend primarily on identity of objects, not cardinality.

(10) a. *Different people like different things* is true iff at least two people like something, and for any two such people, the things one likes are not exactly the same as those the second likes.

b. (DIFF-DIFF)$(A,B)(R) = 1$ iff $|\mathsf{Dom}(R \cap (A \times B))| \geq 2$ and $\forall a \neq a' \in \mathsf{Dom}(R \cap (A \times B))$

$$\{y \in B \mid (a,y) \in R \cap (A \times B)\} \neq \{y \in B \mid (a',y) \in R \cap (A \times B)\}.$$

An example which moves away from the Determiner class of expressions are reflexives, (11a), and reciprocals, (11b):

(11) a. Every senator admires *himself*,
b. The candidates criticized *each other*.

himself and *each other* contrast with non-AI Noun Phrases (NPs) like *the President* in *Every senator admires the President* and *the reporters* in *The candidates criticized the reporters*. We treat *himself* as denoting that function SELF from binary relations (e.g., ADMIRE) to properties given by: $\text{SELF}(R) = \{a \in \text{E} \mid \langle a, a\rangle \in R\}$. Clearly SELF is AI. And we interpret *each other* as that AI map EO from binary relations to maps from sets to truth values given (on first approximation) by: $\text{EO}(R)(A) = 1$ iff $\forall a \neq a' \in A, aRa'$.

A last, non-quantificational, example: Arguably the P_1 *is opened* in (12b) is derived by a logical operation, Passive, from the P_2 *opens* mapping binary relations to sets: $\text{PASS}(R) = \{b \in \text{E} \mid \exists a \in \text{E}\ aRb\} = \text{Ran}(R)$. Obviously PASS is AI. In (12c) the role of the preposition *by* is to say that its object, *John*, is the "logical subject" of the P_2 *opens*. It combines with NPs to yield binary relation modifiers, and so has type $((p,t),(r,r))$.

(12) a. John opens the door every morning at nine,
b. The door is opened every morning at nine,
c. The door is opened by John every morning at nine.

Keenan (*1980*) defines: $\text{BY}(z)(R)(y)(x) = 1$ iff $\langle x, y\rangle \in R$ and $x = z$. Analyzing (12c) as (the door)(Pass(by John)(open))) yields correct truth conditions. And clearly $\text{BY} \in \text{AI}_\Omega((p,t),(r,r))$.

2.1. Logical constants

We have found a property, that of being AI, which classical logical constants have in common. And we have shown that a variety of novel expressions which are pretheoretically "logical" also have this property. This would seem to refute Etchemendy (*1990*, chapter 15 "The Myth of the Logical Constant", page 128) who claims that there is no property of expressions whose interpretations we hold fixed that accounts for their role in establishing logical truths and valid inferences. But denoting AI elements in their type is a good candidate. In fact:

THEOREM 17 (Keenan and Stavi, *1986*). *Constantly interpreted expressions denote* AI *objects.*

We extend the theorem to type theoretic $\mathcal{L}$s, rich enough to include logical expressions of the sort discussed above. And we define *constantly interpreted* for these $\mathcal{L}$s.

Syntax. Type theoretic $\mathcal{L}$s with equality have, for each type τ, countably many (possibly zero) constants Con_τ and denumerably many variables with

$\mathsf{Var}_\tau \cap \mathsf{Var}_{\tau'} = \varnothing$ if $\tau \neq \tau'$. If d and d' are expressions of the same type, then $(d = d')$ is an expression of type t. Expressions of type (α, β) concatenate with ones of type α to form ones of type β. If $d_1, \ldots, d_k$ are expressions of types $\tau_1, \ldots, \tau_k$ respectively, then $\langle d_1, \ldots, d_k \rangle$ is an expression of type $(\tau_1 \bullet \ldots \bullet \tau_k)$. Boolean compounding, universal (existential) quantification apply just to expressions of boolean types.

Semantics. $\mathrm{M} = (\mathrm{E_M}, 2_\mathrm{M}, \mathcal{I})$ is a model for a type theoretic $\mathcal{L}$ iff $(\mathrm{E_M}, 2_\mathrm{M})$ is an ontology and $\mathcal{I}$ is a function mapping each $c \in \mathsf{Con}_\tau$ into $\mathrm{D}_{(\mathrm{E},2)}(\tau)$, all $\tau \in$ *type* (subscripts omitted). An *assignment* s maps each $x \in \mathsf{Var}_\tau$ into $\mathrm{D}_{(\mathrm{E},2)}\tau$. An *interpretation* $\mathcal{I}_\mathrm{M}$ *of* $\mathcal{L}$ *relative to* M maps $\mathcal{L}$-expressions to functions from assignments into the $\mathrm{D}_{(\mathrm{E},2)}\tau$ such that: For $c \in \mathsf{Con}_\tau$, $\mathcal{I}_\mathrm{M}(c)(s) = \mathcal{I}(c)$. For $v \in \mathsf{Var}_\tau, \mathcal{I}_\mathrm{M}(v)(s) = s(v)$. For d of type (α, β) and b of type α, $\mathcal{I}_\mathrm{M}(d \frown b)(s) =_{\mathrm{df}} \mathcal{I}_\mathrm{M}(d)(s)(\mathcal{I}_\mathrm{M}(b)(s))$. $\mathcal{I}_\mathrm{M}\langle d_1, \ldots, d_k\rangle(s) = \langle \mathcal{I}_\mathrm{M}(d_1)(s), \ldots, \mathcal{I}_\mathrm{M}(d_k)(s)\rangle$. $\mathcal{I}_\mathrm{M}(d = d')(s) = 1$ iff $\mathcal{I}_\mathrm{M}(d)(s) = \mathcal{I}_\mathrm{M}(d')(s)$. Boolean compounds are interpreted pointwise on the assignments, and for each $x \in \mathsf{Var}_\tau$ and each expression ϕ of boolean type, $\mathcal{I}_\mathrm{M}(\exists x \phi)(s) = \bigvee\{\mathcal{I}_\mathrm{M}(\phi)[s/a] \mid a \in \mathrm{D}_{(\mathrm{E},2)}(\tau)\}$. (Each boolean $\mathrm{D}_{(\mathrm{E},2)}\tau$ is complete.) Note: $\mathcal{L}$-models are closed under isomorphism: if $\mathrm{M} = (\mathrm{E}, 2, \mathcal{I})$ is a model and π an automorphism of E and of 2, then $\pi\mathrm{M} =_{\mathrm{df}} (\pi\mathrm{E}, \pi 2, \pi \circ \mathcal{I})$ is an $\mathcal{L}$-model.

For d an expression of $\mathcal{L}$ and M a model M, when $\mathcal{I}_\mathrm{M}(d)$ is constant on the assignments we write $\mathcal{I}_\mathrm{M}(d)$ for $\mathcal{I}_\mathrm{M}(d)(s)$, s any assignment. If d is *closed* (= no free variables), $\mathcal{I}_\mathrm{M}(d)$ is constant on the assignments. In what follows we need Theorem 20, the appropriate generalization of $\mathrm{M} \simeq \mathrm{M}' \Rightarrow \mathrm{M} \cong \mathrm{M}'$; the proof uses the technical Lemmas 18 and 19. '$\mathcal{L}$' ranges over type theoretic languages.

LEMMA 18. *Given* $\mathrm{M} = (\mathrm{E}, I)$, π *a bijection with domain* E, s *an assignment,* $x \in \mathsf{Var}_\tau$, $a \in \mathrm{D}_\Omega\tau$ *and writing* πs *for* $\pi \circ s$, *we see that* $\pi(s[x/a]) = (\pi s)(x/\pi a)$. *Moreover,* $A = B$, *where*

$$A = \{\mathcal{I}_{\pi\mathrm{M}}(d)((\pi s)[x/a])) \mid a \in \mathrm{D}_\Omega\tau\} \text{ and}$$
$$B = \{\mathcal{I}_{\pi\mathrm{M}}(d)((\pi s)(x/\pi a)) \mid a \in \mathrm{D}_\Omega\tau\}.$$

Proof.

(1) $A \subseteq B$. Let $\delta \in A$. Then for some $a \in \mathrm{D}_\Omega\tau$, $\delta = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s)[x/a]$. Since $\pi \restriction \mathrm{D}_\Omega\tau$ is onto, $a = \pi a'$ for some $a' \in \mathrm{D}_\Omega\tau$, so $\delta = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s)[x/\pi a']$, so $\delta \in B$.

(2) $B \subseteq A$. Let $\delta \in B$. Then for some $a \in \mathrm{D}_\Omega\tau$, $\delta = \mathcal{I}_{\pi\mathrm{M}}(d)((\pi s)[x/\pi a])$. Hence for some $b \in \mathrm{D}_\Omega\tau$, $\delta = \mathcal{I}_{\pi\mathrm{M}}(d)((\pi s)[x/b])$, so $\delta \in A$. ⊣

LEMMA 19. *For* $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ *a model for* $\mathcal{L}$, π *a bijection with domain* E, *and* s *an assignment,*

$$\pi(\mathcal{I}_\mathrm{M}(d)(s)) = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s), \qquad \textit{for all expressions } d \textit{ of } \mathcal{L}.$$

We give the proof in full in the Appendix.

THEOREM 20. *For type theoretic $\mathcal{L}$s, Model Isomorphism $\Rightarrow$ Elementary Equivalence.*

Proof. Let $\pi : \mathrm{M} \simeq \mathrm{M}'$, $\phi \in \mathsf{Sent}(\mathcal{L})$, s any M-assignment. Then,

$\mathcal{I}_{\mathrm{M}}\phi = \mathcal{I}_{\mathrm{M}}\phi(s) = \pi(\mathcal{I}_{\mathrm{M}}\phi(s))$, π fixes truth values, $= \mathcal{I}_{\mathrm{M}'}\phi(\pi s)$, Lemma 19, $= \mathcal{I}_{\mathrm{M}'}\phi$ since ϕ is closed. ⊣

DEFINITION 8. An expression d of $\mathcal{L}$ is *constant* iff for all models M, $\mathcal{I}_{\mathrm{M}}d$ is constant on the assignments and for all models M′, all bijections $\pi : \mathrm{E}_{\mathrm{M}} \rightarrow \mathrm{E}_{\mathrm{M}'}$, $\pi(\mathcal{I}_{\mathrm{M}}(d)) = \mathcal{I}_{\mathrm{M}'}(d)$.

THEOREM 21. *A closed expression d of type τ is constant*

$$\Rightarrow \forall \mathrm{M} = (\mathrm{E}, 2, \mathcal{I}), \mathcal{I}_{\mathrm{M}}(d) \in \mathrm{AI}_{(\mathrm{E},2)}\tau.$$

Proof. Given $\mathrm{M} = (\mathrm{E}, 2, \mathcal{I})$, let $\pi \in \mathrm{PERM}(\mathrm{E}_{\mathrm{M}})$. Show $\mathcal{I}_{\mathrm{M}}(d) = \pi(\mathcal{I}_{\mathrm{M}}(d))$. Let s an arbitrary assignment. Then $\pi(\mathcal{I}_{\mathrm{M}}(d)) = \pi(\mathcal{I}_{\mathrm{M}}(d)(s)) = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s)$, Lemma 18, $= \mathcal{I}_{\mathrm{M}}(d)(\pi s)$ since $\mathrm{E}_{\mathrm{M}} = \mathrm{E}_{\pi\mathrm{M}}$ and d is locally constant, $= \mathcal{I}_{\mathrm{M}}(d)(s)$ since $\mathcal{I}_{\mathrm{M}}d$ is constant on the assignments, $= \mathcal{I}_{\mathrm{M}}(d)$. ⊣

THESIS 1. The logical constants in a type theoretic $\mathcal{L}$ are the constants as defined in (8).

Logical constants like '=', 'and', 'not', and '∀' are in fact constant. Some might want to require that logical constants be syntactically simple, though this does not seem essential and would have the unpleasant consequence that a defined simple expression could be logically constant even though the complex defining statement was not. (E.g., let *Taut* abbreviate $\forall x, y(x = y \text{ or } x \neq y)$). Also Westerståhl (*1985*), van Benthem (*1989a*), Keenan and Stavi (*1986*) and Sher (*1991*) note that Definition 8 allows as logical constants expressions defined by AI conditions, like the hypothetical *lottsa* below, where for all models $\mathrm{M} = (\mathrm{E}, \mathcal{I})$,

$$(13)\ \mathcal{I}(\text{lottsa}) = \begin{cases} \text{ALL} & \text{if E is finite,} \\ \text{INFINITELY MANY} & \text{if E is infinite.} \end{cases}$$

We stress that Theorem 21 and Thesis 1 assume closure under isomorphism for the class of models of the $\mathcal{L}$s in question. Here for example is an attempt to construct a constant whose denotation is not AI: Let *blik* be an expression of type (e, t) and suppose that the definition of interpretation requires that for $\mathrm{M} = (\mathrm{E}, 2, \mathcal{I}), \mathcal{I}(\text{blik}) = \{3\}$ if $3 \in \mathrm{E}$, otherwise $\mathcal{I}(\text{blik}) = \mathrm{E}$. *Blik* is constantly interpreted but $\mathcal{I}(\text{blik})$ is not AI when $\mathrm{E} = \{3, 4\}$. But this condition violates closure under isomorphism. If π is a bijection: $\{3, 4\} \rightarrow \{5, 6\}$ then $\pi\mathrm{M} = (\{5, 6\}, \pi 2, \pi \circ \mathcal{I})$ is not a model of $\mathcal{L}$ since $(\pi \circ \mathcal{I})(\text{blik}) = \{\pi 3\}$ is a unit set, but the definition of interpretation requires it to be $\{5, 6\}$.

Note that $\mathcal{L}$s may present "logical" expressions which are not constant. Candidates are Dets like *several* and *a few* ($\neq$ *many* and *few*). *Several* seems vague. Just how many sparrows on a clotheline count as several? At least two, and perhaps on some occasions that suffices. One account of this vagueness (*Keenan and Stavi 1986*) would merely require of models of English that $\mathcal{I}$(several) $\in$ {AT LEAST $n \mid 2 \leq n$}. Then we would have models with the same universe in which *several* was interpreted differently, and so not constantly interpreted, but its denotation would always be AI in its type. We turn now to the central thesis of this article:

3. THESIS 2

THESIS 2. For all ontologies Ω and all types τ, the *logical* elements of $\mathrm{Den}_\Omega\tau$ are the AI ones.

3.1.

Empirical support for Thesis 2 is given by Theorem 21: given an ontology Ω, $\mathrm{AI}_\Omega\tau$ provides the denotations of logical constants of type τ. And this observation generalizes. The constants considered so far are ones that, given an ontology Ω, uniquely determine an element of some $\mathrm{Den}_\Omega\tau$. But many expressions, including ones already considered, determine objects in multiple types, where, in each type, the object is AI. Here are some examples:

(14) *and, or, neither...nor* ... combine in English with expressions in most boolean types to form expressions in that type:

VerbPhrases: John both laughed and cried; neither laughed nor cried.

Transitive VPs: John both hugged and kissed / neither hugged nor kissed his cat.

Adjectives: a smart and industrious / neither smart nor industrious student.

Prepositions: John lives either in or near / neither in nor near New York City.

FACT 4. For all Ω, all boolean τ, AND_τ of type $((\tau \bullet \tau), \tau)$ which maps each (a, b) to $a \wedge_\tau b$, the greatest lower bound of $\{a, b\}$ in $\mathrm{D}_\Omega\tau$, is in $\mathrm{AI}_\Omega((\tau \bullet \tau), \tau)$.

And needless to say, all booleanly definable constants, functions, and relations determine objects which are logical in the appropriate type. Here are some additional AI functors:

(15) a. Typed Equality: For all Ω, all τ, $=_\tau$ of type $((\tau \bullet \tau), t)$ given by: $=_\tau (a, b) = 1_2$ iff $a = b$, is in $\mathrm{AI}_\Omega((\tau \bullet \tau), t)$

b. Typed Membership: For all Ω, all τ, $\in_\tau$, of type $((\tau \bullet (\tau, t)), t)$ which sends each (a, g) in its domain to $g(a)$ is in $\mathrm{AI}_\Omega((\tau \bullet (\tau, t)), t)$, as is $\not\in_\tau$, defined in the obvious way.

Note that $\in_\tau$ is the special case of $\mathrm{APPLY}_{\tau,\sigma}$ with $\sigma = t$:

c. Typed Application: For all Ω, all types τ, σ, $\mathrm{APPLY}_{\tau,\sigma}$ is that element of $\mathrm{D}_\Omega((\tau \bullet (\tau, \sigma)), \sigma)$ mapping each (a, g) in its domain to $g(a)$. It is in $\mathrm{AI}_\Omega((\tau \bullet (\tau, \sigma)), \sigma)$.

(15b,c) characterize the sense in which set membership is a "logical" notion. Note that in a $\mathcal{L}$ in which '$\in$' is a P_2 of type $((e \cdot e), t)$ interpreted as set membership, its denotation is not in general $\mathrm{AI}_\Omega((e \cdot e), t)$. The only AI binary relations over E are $\emptyset$, $\mathrm{E} \times \mathrm{E}$, $\mathrm{ID_E} = \{\langle a, a\rangle \mid a \in \mathrm{E}\}$, and $\neg \mathrm{ID_E} = \{\langle a, b\rangle \mid a, b \in \mathrm{E}\ \&\ a \neq b\}$; see Theorem 28. Set membership fails to coincide with any of these relations except when no x in E is a member of any y in E.

Other "transcendental" (*van Benthem 1989a*) functors like Composition and Substitution, so useful in linguistic description, are also easily seen to be AI in the appropriate type.

d. $\mathrm{COMP}_{\sigma,\tau,\upsilon}$ in $\mathrm{D}_\Omega(((\sigma, \tau) \bullet (\tau, \upsilon)), (\sigma, \upsilon))$ mapping $\langle f, g\rangle$ to $g \circ f$, is in $\mathrm{AI}_\Omega(((\sigma, \tau)(\tau, \upsilon))(\sigma, \upsilon))$

e. $\mathrm{SUB}_{\sigma,\tau,\upsilon}$, that element S of $\mathrm{D}_\Omega(\sigma, ((\sigma, (\upsilon, \tau)), (\sigma, \upsilon)))$ such that $S(a)(f)(g) = f(a)(g(a))$, is in $\mathrm{AI}_\Omega((\sigma, ((\sigma, \tau), (\sigma, \upsilon)))$.

Finally, observe that the property of being "logical" is itself "logical". That is, without circularity, the property of being AI is itself AI:

THEOREM 22 **"Being logical is logical"**. *For all ontologies* $\Omega = (\mathrm{E}, 2)$ *and all types* τ, *that element* AI_τ *of* $\mathrm{Den}_\Omega(\tau, t)$ *which maps each* d *in* $\mathrm{Den}_\Omega\tau$ *to* 1_2 *iff* $d \in \mathrm{AI}_\Omega\tau$, *is in* $\mathrm{AI}_\Omega(\tau, t)$.

Proof. We sketch the direct proof by cases to confirm the absence of paradox. Let Ω be an arbitrary ontology, τ an arbitrary type. Either τ is 'e', 't', a functional type or a product type. We show that in each case the function AI_τ is AI (that is, is an element of $\mathrm{AI}_\Omega(\tau, t)$). In what follows we often write "d is AI in type τ" for $d \in \mathrm{AI}_\Omega\tau$. Also we use repeatedly that an object is AI in a type iff its image under any automorphism is AI in that type.

$\tau = t$. Each element (omitting subscripts) 0,1 of $\mathrm{Den}_\Omega t$ is AI, so AI_t maps each to 1. We must show that $\pi(\mathrm{AI}_t) = \mathrm{AI}_t$, for each Ω automorphism π. Now $\pi(\mathrm{AI}_t)(x) = \pi(\mathrm{AI}_t(\pi^{-1}x)) = \pi(1) = 1$, so $\pi(\mathrm{AI}_t) = \mathrm{AI}_t$.

$\tau = e$. If $\mathrm{Den}_\Omega e$ has just one element, b, then b is AI so $\mathrm{AI}_e(b) = 1$. And $\pi(\mathrm{AI}_e)(b) = \pi(\mathrm{AI}_e(\pi^{-1}(b))) = \pi(\mathrm{AI}_e(b)) = \pi(1) = 1$, so $\pi(\mathrm{AI}_e) = \mathrm{AI}_e$. If

$\text{Den}_\Omega e$ has more than one element, then for each $x \in \text{Den}_\Omega e$ x is not AI, so $\text{AI}_e(x) = 0$ and $\pi(\text{AI}_e)(x) = \pi(\text{AI}_e(\pi^{-1}(x)) = \pi(0) = 0$, so $\pi(\text{AI}_e) = \text{AI}_e$.

τ is a functional type (a, b). We must show that $\text{AI}_{(a,b)} = \pi(\text{AI}_{(a,b)})$. Assume first that $g \in \text{Den}_\Omega a$ is AI, so $\text{AI}_{(a,b)}(g) = 1$. Then $\pi(\text{AI}_{(a,b)})(g) = \pi(\text{AI}_{(a,b)})(\pi g) = \pi(\text{AI}_{(a,b)}(g)) = \pi(1) = 1$, so the two functions take the same values at AI arguments. Now assume g is not AI. So $\text{AI}_{(a,b)}(g) = 0$. And $\pi(\text{AI}_{(a,b)})(g) = \pi(\text{AI}_{(a,b)}(\pi^{-1}g)) = \pi(0) = 0$. Thus $\text{AI}_{(a,b)}$ and $\pi(\text{AI}_{(a,b)})$ take the same values at all g and so are the same function.

τ is a product type $a = a_1 \cdot \ldots \cdot a_n$. Let $x \in \text{Den}_\Omega a$. Observe that x is AI in its product type iff each x_i is AI in its type a_i. If x is AI then $\text{AI}_a(x) = 1$. And $\pi(\text{AI}_a)(x) = \pi(\text{AI}_a(\pi^{-1}(x))) = \pi(\text{AI}_a(x)) = \pi(1) = 1$, so equality holds in this case. If x is not AI then $\text{AI}_a(x) = 0$ and $\pi(\text{AI}_a)(x) = \pi(\text{AI}_a(\pi^{-1}(x))) = \pi(0) = 0$, completing the proof. ⊣

3.2. Plausibility

Granted that the technically defined $\text{AI}_\Omega\tau$ provide denotations of logical expressions, why should we expect this be so? We offer answers from two perspectives.

1. Regarding our type theoretical primitives, the elements of 2 are given as logical: 1_2 is uniquely the value a conjunction of sentences has iff each conjunct has that value, and 0_2 is uniquely the (distinct) value assigned to the negation of a sentence with value 1_2. The permutations we study do not change these logical objects. So to some extent we are asking what other objects must not change as a result of holding the truth values constant? These are ones whose "logicality" follows from the logicality of the truth values.

E, however, is randomly chosen; its elements do not in general have any logical character. But the *permutations* of E do. They are, in effect, the functions which fully respect the boolean structure of properties of elements of E, that is, they are the boolean automorphisms of P_{E}. Writing $\text{ATOM}(P_{\text{E}})$ for the set of unit sets in P_{E}, observe:

FACT 5.

(a) If δ is a boolean automorphism of P_{E}, then $\delta \restriction \text{ATOM}(P_{\text{E}})$ is a permutation of $\text{ATOM}(P_{\text{E}})$. So δ determines $\delta' \in \text{PERM}(\text{E})$, where $\delta'(x) = y$ iff $\delta\{x\} = \{y\}$.

(b) Conversely each permutation π of E uniquely determines a boolean automorphism π^* of P_{E} by: $\pi^*(\text{K}) = \{\pi(k) \mid k \in \text{K}\}$.

To say that a permutation π of E preserves the boolean structure of P_{E} is to say informally that π preserves the "and, or, and not" structure of P_{E}. E.g., π(LAUGH OR CRY) = π(LAUGH) $\cup$ π(CRY); π(NEITHER LAUGH

NOR CRY) = $\neg(\pi(\text{LAUGH}) \cup \pi(\text{CRY}))$. π also preserves the universal and existential quantificational structure of P_E. Note that the set denoted by *praise every boy* is the intersection of the sets denoted by *praise* x taken over all boys x. And π(PRAISE EVERY BOY) = $\pi(\bigcap_{X \in \text{BOY}} \text{PRAISE } X) = \bigcap_{X \in \text{BOY}} \pi(\text{PRAISE } X)$. Similarly π(PRAISE SOME BOY) = $\pi(\bigcup_{X \in \text{BOY}} \text{PRAISE } X) = \bigcup_{X \in \text{BOY}} \pi(\text{PRAISE } X)$.

Finally, as a bijection, a permutation π also respects distinctness of individuals: π does not map distinct individuals to the same one, and π does not omit any individuals. This guarantees that π maps the identity relation to itself. Thus,

(16) *AI objects are ones that remain invariant under substitutions which don't change truth, boolean structure, or identity.*

That AI objects have a logical character then is not surprising.

2. But the intuition behind using AIness to characterize logicality is not that permutations fix properties we pre-theoretically consider logical—most of these observations are theorems, i.e., derived. Rather, our intuition concerns the *generality-specificity* dimension: purely logical relations cannot discriminate one object from another but must treat them uniformally and thus must remain unchanged under systematic substitutions of individuals. One way this uniform treatment shows up is in definability properties. For example, consider definability of n-ary relations over the universe of models for first order $\mathcal{L}$s. Given $\mathcal{L}$ and a model $M = (E, 2, \mathcal{I})$ for $\mathcal{L}$, we say that $R \supseteq E^k$ is $\mathcal{L}$-definable (in M) iff for some $\mathcal{L}$-formula ϕ with exactly the k distinct $x_1, \ldots, x_k$ free, $R = \{t \in E^k \mid M \models \phi[t]\}$, where $\phi[t]$ is $\phi[g]$, $g : \text{VAR} \rightarrow E$ such that $g(x_i) = t_i$, all $1 \leq i \leq k$. Then, recalling that $\mathcal{L}_=$ is the zero signature first order $\mathcal{L}$ with equality,

THEOREM 23. *An n-ary relation R over* E *is* AI *iff R is definable in $\mathcal{L}_=$.*

Proof. Appendix. ⊣

Note that in distinction to definability in higher types (*Läuchli 1970*; *Plotkin 1980*; *van Benthem 1989a*, page 330) Theorem 23 does not require that the universe be finite. No matter how large E is, the AI n-ary relations over E are finite in number (Theorem 28), so the number of such relations of all arities never exceeds the number of expressions in $\mathcal{L}_=$.

COROLLARY 24. *For $R \subseteq E^n$, R is* AI *iff for all first order $\mathcal{L}$ with =, R is definable in all $\mathcal{L}$-models* M *with* $E_M = E$.

Proof. The proof consists in observing that R is preserved under language and model expansions, and any first order $\mathcal{L}$ with = is an expansion of $\mathcal{L}_=$. ⊣

Clearly any relation R definable in $\mathcal{L}_=$ is "logical", as all we can use to define R is equality, boolean operators, and universal and existential quantification. The fact that all AI n-ary relations over an arbitrary universe are definable in $\mathcal{L}_=$ then says that we cannot limit the "logical" relations to a proper subset of the AI ones unless we are prepared to accept that non-logical objects can be defined in terms of logical ones (equality) by logical operations. And a much more general result given by van Benthem (*1989a*) is:

(17) All closed terms in the theory of types define PI objects in their type. Further, in a type hierarchy built from a finite E, every PI item in any type is definable by some closed type-theoretic term of that type.

(Type-theoretic terms here differ from expressions in the type-theoretic $\mathcal{L}$s discussed earlier in that (1) they lack constants of arbitrary type, and (2) they may be built by lambda abstraction.) Our last result here concerns *uniform definability*. First,

DEFINITION 9. Let $\tau = \tau_1 \bullet \ldots \bullet \tau_k$ be a product type and R be a functor of type (τ, t), i.e., R maps each $\Omega = (\mathrm{E}, 2)$ into $\mathrm{Den}_\Omega(\tau, t)$. (So $R(\Omega) \subseteq \mathrm{D}_\Omega\tau_1 \times \ldots \times \mathrm{D}_\Omega\tau_k$.) Then for all languages $\mathcal{L}$, R is *uniformly definable* in $\mathcal{L}$ iff for some $v = \langle v_1, \ldots, v_k\rangle$ of k distinct variables of types $\tau_1, \ldots, \tau_k$ respectively, for some $\mathcal{L}$-formula ϕ free for v and for all models $\mathrm{M} = (\mathrm{E}, 2, \mathcal{I})$ for $\mathcal{L}$, $\mathrm{M} \models \phi[s]$ iff $\langle sv_1, \ldots, sv_k\rangle \in R(\Omega)$.

THEOREM 25. *$\forall\mathcal{L}$, $\forall$functors R of type (τ, t), R is uniformly definable in $\mathcal{L} \Rightarrow R(\Omega)$ is* AI, *all Ω.*

Proof. Let R be uniformally definable in $\mathcal{L}$ by ϕ. Let $\Omega = (\mathrm{E}, 2)$ arbitrary, π an automorphism of Ω. Show $R(\Omega) = \pi(R(\Omega))$.

$\Rightarrow$ Let $u \in R(\Omega)$. By uniform definability $\exists \mathrm{M}$, $\mathrm{M} \models \phi[u]$. So $\pi^{-1}\mathrm{M} \models \phi[\pi^{-1}u]$ by Lemma 19 since π^{-1} is an automorphism, so $\pi^{-1}u \in R(\Omega)$, whence $u \in \pi(R(\Omega))$.

$\Leftarrow$ Let $u \in \pi(R(\Omega))$. So $u = \pi t$ for some $t \in R(\Omega)$. Thus $\exists \mathrm{M}'$, $\mathrm{M}' \models \phi[t]$. So $\pi\mathrm{M}' \models \phi[\pi t]$ by Lemma 19, so by uniform definability $\pi t = u \in R(\pi\Omega) = R(\Omega)$ since π is an automorphism. ⊣

These results hold because AIness is preserved by the operations in the definitions:

THEOREM 26. *For $\Omega = (\mathrm{E}, 2)$ given,*

(1) *The type forming operations preserve* AI*ness: if A and B are sets of* AI *objects then all elements of $[A \to B]$, $A \times B$ and of course $\mathcal{P}(A) \simeq [A \to 2]$ are* AI.

(2) *For each boolean τ, $\mathrm{AI}_\Omega\tau$ is a (complete, atomic) boolean subalgebra of $\mathrm{Den}_\Omega\tau$. In other words, boolean functions preserve* AI*ness.*

(3) *Generalized converses: If R is a* AI *k-ary relation and* γ *is a permutation of* $\{1,\ldots,k\}$, *then* $\gamma R =_{\mathrm{df}} \{\langle a_{\gamma(1)},\ldots,a_{\gamma(k)}\rangle \mid \langle a_1,\ldots,a_k\rangle R\}$ *is a* AI *k-ary relation.*

(4) *Cylindrification (existential quantification): If R is a k+1-ary* AI *relation, then for all* $1 \leq i \leq k$, $C_i(R) =_{\mathrm{df}} \{\langle a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1}\rangle \mid \exists a \in A_i\langle a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_{k+1}\rangle \in R\}$ *is* AI.

(5) *The value of a* AI *function at a* AI *argument is itself* AI. *As a special case, composition of* AI *relations (functions) yields a* AI *relation (function).*

3.3 Additional conditions? Granting the logical objects are AI ones, are we ready to grant that all AI objects (of a given type, in a given ontology) are "logical"? Our remarks above on definability suggest that we do, pending plausible additional conditions. Here are two suggestions, the first of which is only apparent, the second of which rules out too much.

3.3.1 The notion of AIness generalizes naturally to invariance under isomorphisms (ISOM) and trivially objects which are ISOM are also AI. In fact equality holds, so the apparently stronger requirement does not in fact eliminate any candidates. We sketch the result.

First, a map π with domain *type* is an *isomorphism* from an ontology $\Omega = (\mathrm{E}, 2)$ to an ontology $\Omega' = (\mathrm{E}', 2')$ iff for all initial types τ, $\pi(\tau)$ is an isomorphism: $\mathrm{Den}_{\Omega}\tau \simeq \mathrm{Den}_{\Omega'}\tau$ and the value of π at functional and product types lifts just as in the case of automorphisms. An element d of $\mathrm{Den}_{\Omega}\tau$ is said to be *isomorphism invariant* iff for all ontologies Ω', all isomorphisms π, δ from Ω to Ω', $\pi(d) = \delta(d)$. And we prove:

THEOREM 27. *For all* $\Omega = (\mathrm{E}, 2)$, *all* $\tau \in$ *type*, *all* $d \in \mathrm{D}_{\Omega}\tau$, $d \in \mathrm{ISOM}_{\Omega}\tau \Leftrightarrow d \in \mathrm{AI}_{\Omega}\tau$.

Proof. Only $\Leftarrow$ is non-obvious. The proof is by recursion on *type*.

$$\text{Set } \mathrm{K} = \{\tau \in \textit{type} \mid \text{for all } \Omega = (\mathrm{E}, 2), \text{ all } d \in \mathrm{D}_{\Omega}\tau,\ d \in \mathrm{AI}_{\Omega}\tau \Rightarrow d \in \mathrm{ISOM}_{\Omega}\tau\}.$$

i. Obviously $t \in \mathrm{K}$ since for any $\Omega' = (\mathrm{E}', 2')$ there is only one isomorphism: $2 \to 2'$. And $e \in \mathrm{K}$ since if $\mathrm{AI}_{\Omega}e$ is non-empty, then E has just one element, whence $|\mathrm{E}'| = 1$ so again there is only one bijection from E to E'.

ii. Let $\alpha, \beta \in \mathrm{K}$, show $(\alpha, \beta) \in \mathrm{K}$. Let $d \in \mathrm{D}_{\Omega}(\alpha, \beta)$, let π, δ be isomorphisms from Ω to Ω'. Then $\pi_{(\alpha,\beta)}(d)(\pi_{\alpha}(a)) = \pi_{\beta}(d(a))$, by def ISOM, $= \delta_{\beta}(d(a))$ by the IH, $= \delta_{(\alpha,\beta)}(d)(\delta_{\alpha}(a)) = \delta_{(\alpha,\beta)}(d)(\pi_{\alpha}(a))$ by IH, so $\pi_{(\alpha,\beta)}(d) = \delta_{(\alpha,\beta)}(d)$, so $(\alpha, \beta) \in \mathrm{K}$.

iii. Closure under products is straightforward. $\dashv$

3.2 Universe independence? One might have hoped for a more "absolute" notion of logical object, one that did not depend on the choice of universe or type. Note that on the view presented here, FACT 6. For each ontology Ω and type τ, $\mathsf{Den}_\Omega\tau$ is in $\mathrm{AI}_\Omega(\tau, t)$. It is in fact its boolean unit.

So each element in the type hierarchy built on an $(\mathrm{E}, 2)$ is AI in a certain type relative to that ontology. But it doesn't make sense to claim that, e.g., $\{3,4\}$ is "absolutely" PI. It is in $\mathrm{AI}_{(\mathrm{E},2)}(e,t)$ when $\mathrm{E} = \{3,4\}$ but not otherwise. Similarly $\{\langle 2,2\rangle, \langle 3,3\rangle\}$ is AI over the universe $\mathrm{E} = \{2,3\}$ but fails to be AI over any other universe, say $\{1,2,3\}$ or $\{1,2\}$. More generally, for $\Omega = (\mathrm{E}, 2)$ and $\Omega' = (\mathrm{E}', 2')$ with E, E' disjoint and $2, 2'$ disjoint, we have that for all τ, $\mathrm{D}_\Omega\tau$ and $\mathrm{D}_{\mathrm{E}'}\tau$ are disjoint. Thus the attempt to characterize invariants independent of ontology seems bootless. Nor can we omit mention of type. An object d in some $\mathrm{AI}_\Omega\tau$ fails to be in $\mathrm{AI}_{\Omega'}(e)$, where the universe of Ω' includes $\{d, \langle d, 0\rangle, \langle d, 1\rangle\}$. And we have thus shown:

FACT 7. For d in any $\mathrm{AI}_\Omega\tau$ there is an ontology Ω' and type τ' such that $d \in \mathsf{Den}_\Omega\tau - \mathrm{AI}_\Omega\tau$.

4. AN OPEN PROBLEM

Van Benthem (*1989a*) notes as open the problem of characterizing the AI elements of the type hierarchy built from an arbitrary ontology $(\mathrm{E}, 2)$, an obvious desideratum given the close association between AIness and logicality. We do not have a general answer, but we shall answer two basic cases, and make a few remarks concerning some others. (We tend to write PI instead of AI when dealing with types not involving 't'.)

THEOREM 28. *The* PI *n-ary relations over a universe* E *with at least n elements correspond one-to-one to the sets of equivalence relations over an n-element set.*

Proof sketch. $\mathcal{P}(\mathrm{E}^n)$ is a complete, atomic boolean algebra. Direct calculation shows that PIness is preserved under complements and arbitrary intersections, so the PI n-ary relations form a complete and thus atomic subalgebra of $\mathcal{P}(\mathrm{E}^n)$. To construct its atoms let T be any equivalence relation over $\{1, \ldots, n\}$. Set $R_T = \{s \in \mathrm{E}^n \mid \forall 1 \leq i,j \leq n, s_i = s_j \text{ iff } iTj\}$. Then R_T is AI, non-empty (since $|\mathrm{E}| \geq n$), and no non-empty proper subset of R_T is AI. So R_T is an atom. Clearly different T's give rise to different R'_Ts, and if R is AI and non-empty, then for each $s \in R$, $R_s =_{\mathrm{df}} \{t \in \mathrm{E}^n \mid t_i = t_j \text{ iff } s_i = s_j,$ all $1 \leq i,j \leq n\}$ is an atom which is a subset of R. So the R'_Ts are all the atoms, and since each AI R is the union of its atoms, the AI R's correspond one-to-one to the sets of atoms. Hence $|\{R \in \mathcal{P}(\mathrm{E}^n) \mid R \text{ is AI}\}| = 2^{\mathsf{EQ}(n)}$, where $\mathsf{EQ}(n)$ is the number of equivalence relations over an n element set. ⊣

Westerståhl (*1985*) also gives the figure $2^{\mathsf{EQ}(n)}$ and provides a recursive means of calculating $\mathsf{EQ}(n)$. Our interest in the figure is that the number of AI n-ary relations over E does not grow with the size of (sufficiently large) E but is bounded as a function of the arity of the relations. The same property obtains for the PI functions from E^n into E:

THEOREM 29. *If* $|E| \geq n+2$, *then* $|\{f \in [E^n \to E] \mid f \text{ is PI}\}| = n \cdot \prod_{1 \leq k \leq n-1} (n-k)^{C[n;k+1]}$.

Proof sketch. The PI maps from E^n into E are the projection functions which can choose different coordinates at non-isomorphic tuples, e.g., $\langle a,b,c\rangle$ and $\langle a,b,a\rangle$. If $f \in [E^n \to E]$ is PI, then $\forall \sigma \in E^n f(\sigma) = \sigma_i$ for some $1 \leq i \leq n$, otherwise permuting $f(\sigma)$ with some element not $f(\sigma)$ or any coordinate of σ fails to preserve f. So f restricted to the set of n-tuples with no coordinates identified is one of the n projection functions. For $k > 0$, $C[n; k+1] =_{df} n!/(k+1)! \cdot (n-(k+1))!$ is the number of ways of identifying $k+1$ coordinates of an n-tuple. For each such way the number of different coordinates of the resulting tuple and hence the number of possible values a PI function can take, is $(n-k)$. It takes its values independently on tuples formed by different ways of identifying $k+1$ coordinates. ⊣

So (for E sufficiently large) no $x \in E \simeq [E^0 \to E]$ is PI (so there can be no logically constant proper nouns); only the identity map in $[E \to E]$ is PI, and only the two projection functions in $[E \times E \to E]$ are PI. There are $3 \cdot 2^3 = 24$ PI maps in $[E^3 \to E]$ and $4 \cdot 3^6 \cdot 2^4 = 46,656$ PI maps in $[E^4 \to E]$.

Richer e types seem to behave somewhat erratically. For $|E| > 1$, there are no PI maps in $[[E \to E] \to E]$: Let h be in that set and let π move $h(id)$. Then $(\pi h)(id) = (\pi h)(\pi id) = \pi(h(id)) \neq h(id)$, so h is not PI. This observation is a special case of:

THEOREM 30 (Van Benthem, *1989a*). *For* τ *in the closure of* $\{e,t\}$ *under functional types,* $\mathrm{Den}_\Omega \tau$ *has no* PI *elements iff* $\tau = (\sigma_1, (\sigma_2, \ldots (\sigma_n, e) \ldots))$ *where each* $\mathrm{Den}_\Omega \sigma_i$ *has* PI *elements.*

By contrast in $[[E \to E] \to [E \to E]]$ the number of PI maps grows exponentially in $|E|$, whence for infinite E the number outstrips the number of expressions in the language. Note that different subsets K of $\{0, \ldots, |E|\}$ in (18) give rise to different f_K's, where

(18) $$f_K(g)(a) =_{df} \begin{cases} g(a) \text{ if } |\{a \mid ga = a\}| \in K, \\ a \text{ otherwise.} \end{cases}$$

Similarly, the growth of AI generalized quantifiers is exponential in the size of the universe:

(19) $$|AI[\mathcal{P}(E) \to 2]| = 2^{|E|+1}.$$

But we still lack a general means of computing cardinalities of AI sets in a type hierarchy.

APPENDIX

LEMMA 8. *Let* M, M′ *be infinite models for* $\mathcal{L}_=$. *Then for all formulas* ϕ *in* $\mathcal{L}_=$, *all* $g : \text{VAR} \to E_M$ *and all* $g' : \text{VAR} \to E_{M'}$

$$\text{if } g(x) = g(y) \text{ iff } g'(x) = g'(y), \text{ for all } x, y \in \text{FREEVAR}(\phi),$$
$$\text{then } M \models \phi[g] \text{ iff } M' \models \phi[g'].$$

Proof. By recursion on the formulas of $\mathcal{L}_=$. Let M, M′ be infinite models for $\mathcal{L}_=$.

Set $S = \{\phi \in \mathsf{Fmla}(\mathcal{L}_=) \mid$ for all $g : \text{VAR} \to E_M$ and all $g' : \text{VAR} \to E_{M'}$ if $g(x) = g(y)$ iff $g'(x) = g'(y)$, for all $x, y \in \text{FREEVAR}(\phi)$, then $M \models \phi[g]$ iff $M' \models \phi[g']\}$.

(i) All atomic formulas ϕ are in S. ϕ is $(x = y)$ for some $x, y \in \text{VAR}$. Let g, g' be as in the lemma with $g(x) = g(y)$ iff $g'(x) = g'(y)$. Then $M \models (x = y)[g]$ iff $g(x) = g(y)$, iff $g'(x) = g'(y)$, iff $M' \models (x = y)[g']$, so $(x = y) \in S$.

(ii) S is closed under conjunction and negation.

(a) Let $\phi, \psi \in S$. Show $(\phi \mathbin{\&} \psi) \in S$. Let g, g' be such that $g(x) = g(y)$ iff $g'(x) = g'(y)$, for all x, y free in $(\phi \mathbin{\&} \psi)$. Then for all x, y free in ϕ, $g(x) = g(y)$ iff $g'(x) = g'(y)$ since if, e.g., g identified free variables in ϕ that g' distinguished, then g would have identified them in $(\phi \mathbin{\&} \psi)$ but g' would distinguish them, contrary to hypothesis. Then

$M \models (\phi \mathbin{\&} \psi)[g]$	iff $M \models \phi[g]$ and $M \models \psi[g]$	def interpretation,
	iff $M' \models \phi[g']$ and $M \models \psi[g']$	IH + the observation,
	iff $M' \models (\phi \mathbin{\&} \psi)[g']$	def int.

So $(\phi \mathbin{\&} \psi) \in S$.

(b) Let $\phi \in S$. Show that $\neg\phi \in S$. Note that $\text{FREEVAR}(\neg\phi) = \text{FREEVAR}(\phi)$. So

$M \models \neg\phi[g]$	iff $M \not\models \phi[g]$	def int,
	iff $M' \not\models \phi[g']$	IH + note,
	iff $M' \models \neg\phi[g']$	def int.

So $\neg\phi \in S$.

(iii) S is closed under existential quantification.

Let $\phi \in S$. Show for all variables z, $\exists z\phi \in S$. Let z be arbitrary, let g, g' as in the lemma distinguish the same variables in $\exists z\phi$. Show $M \models \exists z\phi[g]$ iff $M' \models \exists z\phi[g']$.

Left to right. Let $\mathrm{M} \models \exists z\phi[g]$. Then $\mathrm{M} \models \phi[g_z]$, where g_z is a z-variant of g.

case 1: $g_z(z) = g(x)$ for some $x \in \text{FREEVAR}(\exists z\phi)$. Let g'_z be that z-variant of g' such that $g'_z(z) = g'(x)$. Then for all x, y free in ϕ, $g_z(x) = g_z(y)$ iff $g'_z(x) = g'_z(y)$. So by IH, $\mathrm{M}' \models \phi[g'_z]$, whence $\mathrm{M}' \models \exists z\phi[g']$.

case 2: $g_z(z) \neq g(x)$ for any x free in $\exists z\phi$. Then find a z-variant g'_z of g' such that $g'_z(z) \neq g'(x)$, for any x free in ϕ. Such a g'_z exists since $\text{FREEVAR}(\phi)$ is finite and $\mathrm{E}_{\mathrm{M}'}$ is infinite, so there is an object not in the range of $g'_z \restriction \text{FREEVAR}(\phi)$. And again for all $x, y \in \text{FREEVAR}(\phi)$, $g_z(x) = g_z(y)$ iff $g'_z(x) = g'_z(y)$. So by IH, $\mathrm{M}' \models \phi[g'_z]$, whence $\mathrm{M}' \models \exists z\phi[g']$.

This exhausts the cases. The right to left direction is done analogously. ⊣

THEOREM 20. *Model Isomorphism ⇒ Elementary Equivalence for Type Theoretic Languages.*

For $\mathrm{M} = (\mathrm{E}, \mathcal{I})$ a model for $\mathcal{L}$, π a bijection with domain E, s any M-assignment and d an expression of any type in $\mathcal{L}$,

$$\pi(\mathcal{I}_{\mathrm{M}}(d)(s)) = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s).$$

Set $S = \{d \in \mathcal{L} \mid \text{for all assignments } s,\ \pi(\mathcal{I}_{\mathrm{M}}(d)(s)) = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s)\}$. Then,

1. For all τ, $\mathsf{Con}_\tau \subseteq S$. Let $c \in \mathsf{Con}_\tau$. Then,

$\pi(\mathcal{I}_{\mathrm{M}}(c)(s)) = \pi(\mathcal{I}(c))$, def int,
and $\mathcal{I}_{\pi\mathrm{M}}(c)(\pi s) = (\pi \circ \mathcal{I})(c)$, def int, $= \pi(\mathcal{I}(c))$.

2. For all τ, $\mathsf{Var}_\tau \subseteq S$. Let $x \in \mathsf{Var}_\tau$. Then,

$\pi(\mathcal{I}_{\mathrm{M}}(v)(s)) = \pi(s(v))$, def int,
and $\mathcal{I}_{\pi\mathrm{M}}(v)(\pi \circ s) = (\pi \circ s)(v) = \pi(s(v))$.

3. For $d, d' \in S$, d, d' both of type τ. Show $(d = d') \in S$.

$\pi(\mathcal{I}_{\mathrm{M}}(d = d')(s)) = 1$

iff $\mathcal{I}_{\mathrm{M}}(d = d')(s) = 1$	π fixes each $x \in 2$,
iff $\mathcal{I}_{\mathrm{M}}(d)(s) = \mathcal{I}_{\mathrm{M}}(d')(s)$	def interpretation,
iff $\pi(\mathcal{I}_{\mathrm{M}}(d)(s)) = \pi(\mathcal{I}_{\mathrm{M}}(d')(s))$	π is injective,
iff $\mathcal{I}_{\pi\mathrm{M}}(d)(\pi s) = \mathcal{I}_{\pi\mathrm{M}}(d')(\pi s)$	Induction Hypothesis,
iff $\mathcal{I}_{\pi\mathrm{M}}(d = d')(\pi s) = 1$	def interpretation.

4. For $d_1, \ldots, d_k \in S$, show $d = \langle d_1, \ldots, d_k \rangle \in S$.

$$\begin{aligned}
&\pi(\mathcal{I}_{\mathrm{M}}(d)(s) \\
&\quad = \pi(\langle \mathcal{I}_{\mathrm{M}}(d_1)(s), \ldots, \mathcal{I}_{\mathrm{M}}(d_k)(s)\rangle) && \text{def interpretation,} \\
&\quad = \langle \pi(\mathcal{I}_{\mathrm{M}}(d_1)(s)), \ldots, \pi(\mathcal{I}_{\mathrm{M}}(d_k)(s))\rangle && \text{def } \pi \text{ on tuples,} \\
&\quad = \langle \mathcal{I}_{\pi\mathrm{M}}(d_1)(\pi s)), \ldots, \mathcal{I}_{\pi\mathrm{M}}(d_k)(\pi \circ s))\rangle && \text{Induction Hypothesis,} \\
&\quad = \mathcal{I}_{\pi\mathrm{M}}(\langle d_1, \ldots, d_k\rangle)(\pi \circ s) && \text{def interpretation.}
\end{aligned}$$

5. For $d, b \in S$, d of type (α, β) and b of type β, show $d^\frown b \in S$.

$$\begin{aligned}
&\pi(\mathcal{I}_{\mathrm{M}}(d^\frown b)(s)) \\
&\quad = \pi(\mathcal{I}_{\mathrm{M}}(d)(s)(\mathcal{I}_{\mathrm{M}}(b)(s)) && \text{def interpretation,} \\
&\quad = \pi(\mathcal{I}_{\mathrm{M}}(d)(s)(\pi(\mathcal{I}_{\mathrm{M}}(b)(s))) && \text{def } \pi \text{ on function spaces,} \\
&\quad = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi s)(\mathcal{I}_{\pi\mathrm{M}}(b)(\pi \circ s)) && \text{Induction Hypothesis,} \\
&\quad = \mathcal{I}_{\pi\mathrm{M}}(d^\frown b)(\pi \circ s) && \text{def interpretation.}
\end{aligned}$$

6. For $d, d' \in S$, d, d' of boolean type τ, show $(d \;\&\; d') \in S$ and $\neg d \in S$.

$$\begin{aligned}
&\pi(\mathcal{I}_{\mathrm{M}}(d \;\&\; d')(s)) \\
&\quad = \pi(\mathcal{I}_{\mathrm{M}}(d)(s) \wedge \mathcal{I}_{\mathrm{M}}(d')(s)) && \text{def interpretation,} \\
&\quad = \pi(\mathcal{I}_{\mathrm{M}}(d)(s)) \wedge \pi(\mathcal{I}_{\mathrm{M}}(d')(s)) && \pi \text{ a boolean automorphism,} \\
&\quad = \mathcal{I}_{\pi\mathrm{M}}(d)(\pi \circ s)) \wedge \mathcal{I}_{\pi\mathrm{M}}(d')(\pi \circ s)) && \text{Induction Hypothesis,} \\
&\quad = \mathcal{I}_{\pi\mathrm{M}}(d \;\&\; d')(\pi \circ s) && \text{def interpretation.}
\end{aligned}$$

The proof that S is closed under negation follows the same pattern.

7. For $x \in \mathsf{Var}_\tau$ and ϕ of boolean type σ, assume $\phi \in S$. Show $\exists x\phi \in S$.

$$\begin{aligned}
&\pi(\mathcal{I}_{\mathrm{M}}(\exists x\phi)(s)) \\
&\quad = \pi(\bigvee\{\mathcal{I}_{\mathrm{M}}(\phi)(s[x/a] \mid a \in \mathrm{D}_\Omega\tau\}) && \text{def interpretation,} \\
&\quad = \bigvee\{\pi(\mathcal{I}_{\mathrm{M}}(\phi)(s[x/a] \mid a \in \mathrm{D}_\Omega\tau)\} && \pi \restriction \mathrm{D}_\Omega\tau \text{ a boolean automorphism,} \\
&\quad = \bigvee\{\mathcal{I}_{\pi\mathrm{M}}(\phi)(\pi(s[x/a])) \mid a \in \mathrm{D}_\Omega\tau\} && \text{Induction Hypothesis,} \\
&\quad = \bigvee\{\mathcal{I}_{\pi\mathrm{M}}(\phi)(\pi \circ s)[x/\pi a])) \mid a \in \mathrm{D}_\Omega\tau\} && \text{Lemma 18,} \\
&\quad = \bigvee\{I_{\pi\mathrm{M}}(\phi)(\pi \circ s)[x/a])) \mid a \in \mathrm{D}_\Omega\tau\} && \text{Lemma 18,} \\
&\quad = \mathcal{I}_{\pi\mathrm{M}}(\exists x\phi)(\pi \circ s) && \text{def interpretation.}
\end{aligned}$$

⊣

THEOREM 23. $\Rightarrow$ *The* $\mathcal{L}_=$*-definable* $n \geq 1$*-ary relations over* E *are just the* $R_{\phi,v} =_{\mathrm{df}} \{t \in \mathrm{E}^k \mid \mathrm{M} \models \phi[t]\}$, *any* M *with* $\mathrm{E}_\mathrm{M} = \mathrm{E}$, $v = \langle v_1, \ldots, v_k\rangle$. *We show that each* $R_{\phi,v}$ *is* AI, ϕ *free for* $\{v_i \mid 1 \leq i \leq k\}$. *Set* $S = \{\phi \in \mathcal{L}_= \mid R_{\phi,v}$ *is* AI, *all* v *such that* ϕ *is free for* $v\}$.

Proof.

a. all atomic formulas ϕ are in S.

case 1: ϕ is $(x = y)$ for x, y distinct. Let v be a tuple of distinct variables such that for some i, j $v_i = x$, $v_j = y$. $R_{\phi,v} = \{s \in \mathrm{E}^k \mid \mathrm{M} \models (x = y)[s]\} = \{s \in \mathrm{E}^k \mid s_i = s_j\}$, which is AI.

case 2: ϕ is $(x = x)$. Let v be a tuple of distinct variables with $v_j = x$. Then $R_{\phi,v} = \{s \in \mathrm{E}^k \mid \mathrm{M} \models (x = x)[s_j]\} = \mathrm{E}^k$, again AI.

b. S is closed under conjunction and negation

1. Let $\phi, \psi \in S$. Let $(\phi \;\&\; \psi)$ be free in v, so ϕ, ψ are also free in v. Then

$$R_{\phi\&\psi,v} = \{s \in \mathrm{E}^k \mid \mathrm{M} \models (\phi \;\&\; \psi)[s]\} = \{s \in \mathrm{E}^k \mid \mathrm{M} \models (\phi)[s]\} \cap \{s \in \mathrm{E}^k \mid \mathrm{M} \models (\psi)[s]\},$$
$= R_{\phi,v} \cap R_{\psi,v}$, which is AI since intersection preserves AIness.

2. Let $\phi \in S$. Let $\neg\phi$, and hence also ϕ, be free in v.

$R_{\neg\phi,v} = \{s \in \mathrm{E}^k \mid \mathrm{M} \not\models \phi[s]\} = -R_{\phi,v}$ is again AI since complement preserves AIness.

c. S is closed under existential quantification.

Let $\phi \in S$, let $\exists z\phi$ be free in v. We may assume z is free in ϕ, otherwise $R_{\exists z\phi,v} = R_{\phi,v}$, which is AI. Then $R_{\exists z\phi,v} = \{s \in \mathrm{E}^k \mid \mathrm{M} \models \exists z\phi[s]\} = \{s \in \mathrm{E}^k \mid \exists b \mathrm{M} \models \phi[v_i/s_i, z/b]\}$. If $R_{\exists z\phi,v} = \emptyset$, then it is AI. Set $v' = \langle v_1, \ldots, v_k, z\rangle$. Let $u \in R_{\exists z\phi,v}$, so $\exists b$, $\mathrm{M} \models \phi[s/b]$ and $\langle u, b\rangle \in R_{\phi,v'}$. Let $\pi \in \text{PERM}(\mathrm{E})$. So $\langle \pi u, \pi b\rangle \in R_{\phi,v'}$ by the IH, so $\pi u \in R_{\exists z\phi,v}$, whence $R_{\exists z\phi,v}$ is AI.

$\Leftarrow$ Let $R \subseteq \mathrm{E}^n$ be AI. Show that R is definable in $\mathcal{L}_=$.

i. If $R = \emptyset$, then $R = R_{\phi,v}$ where ϕ is $(x \neq x)$ and $v = \langle x\rangle$.

ii. Let $R \neq \emptyset$. Let $s \in R$. Recall $R_s = \{t \in \mathrm{E}^n \mid t_i = t_j \text{ iff } s_i = s_j, \text{ all } 1 \leq i, j \leq n\}$.

Then fixing the sequence $v = \langle x_1, \ldots, x_n\rangle$ of distinct variables, R_s is defined by the conjunction of $\text{AND}\{(x_i = x_j) \mid s_i = s_j\}$ with $\text{AND}\{(x_i \neq x_j) \mid s_i \neq s_j\}$. Since $R = \bigcup_{s\in R} R_s$, R is defined by the disjunction of the finitely many sentences used to define each R_s. ⊣

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