Further Excursions in Natural Logic: The Mid-Point Theorems[1]

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Pursuing a study begun in (Keenan 2004) this note investigates inference patterns in natural language which proportionality quantifiers enter. We desire to identify such patterns and to isolate any such which are specific to proportionality quantifiers.

Background Keenan (2004) identified the inference pattern in (1) and suggested that it involved proportionality quantifiers in an essential way.

(1) a. More than n/m of the As are Bs
At least 1 – n/m of the As are Cs
Ergo: Some A is both a B and a C

b. At least n/m of the As are Bs
More than 1 – n/m of the As are Cs
Ergo: Some A is both a B and a C

To illustrate (1a): If more than three tenths of the students are athletes and at least seven tenths are vegetarians then at least one student is both an athlete and a vegetarian.

This is indeed a valid argument paradigm. However recently Westerståhl (pc) showed that the pattern (1a,b) is a special case of a more general one not specific to proportionality quantifiers but which includes them simply as a special case. His result supports the claim that proportionality quantifiers enter inference paradigms common to better understood classes of quantifiers. But it also leads us to question whether there are any inference patterns specific to proportionality quantifiers. To pursue these questions we need some background definitions.

Def 1 Given a domain E, the set GQ_E of *generalized quantifiers over* $E =_{def} [P(E) \rightarrow \{0,1\}]$, the set of functions from P(E) into {0,1}. Such functions will also be called (following Lindstrom 1966) functions of type <1>. Interpreting P1s, one place predicates, as elements of P(E) we can use type <1> functions as denotations of the DPs italicized in (2):

(2) a.. *No teacher* laughed at that joke
b. *Every student* came to the party
c. *Most students* are vegetarians

So the truth value of (2a) is the one that the function denoted by *no teacher* assigns to the denotation of the P1 *laughed at that joke*. The Dets *no*, *every*, and *most* combine with a single Noun to form a DP and are naturally interpreted by functions of type <1,1>, namely maps from P(E) into GQ_E. They exemplify three different classes of type <1,1> functions: the **intersective**, the **co-intersective** and the **proportionality** ones. Informally a D of type <1,1> is intersective if its value at sets A,B just depends on $A \cap B$, it is co-intersective if its value depends just on $A - B$,

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and it is proportional if it just depends of the proportion of As that are Bs. Formally, we define these notions and a few others below in terms of invariance conditions.

Def 1 For D of type <1,1>,

a.1 D is *intersective* iff for all sets A,B,X,Y if $A \cap B = X \cap Y$ then DAB = DXY
a.2 D is *cardinal* iff for all sets A,B,X,Y if $|A \cap B| = |X \cap Y|$ then DAB = DXY

b.1 D is *co-intersective* iff for all A,B,X,Y if $A - B = X - Y$ then DAB = DXY
b.2 D is *co-cardinal* iff for all A,B,X,Y if $|A - B| = |X - Y|$ then DAB = DXY

c. D is *proportional* iff for all A,B,X,Y if $|A \cap B|/|A| = |X \cap Y|/|X|$ then DAB = DXY

d. D is *conservative* iff for all A, B, B' if $A \cap B = A \cap B'$ then DAB = DAB'

e. D is *permutation invariant* iff for all $A,B \subseteq E$, all permutations π of E, DAB = $D\pi A\pi B$

One checks that NO, EVERY, and MOST defined below are intersective, co-intersective and proportional respectively. All three of these functions are permutation invariant and conservative.

(3) a. NO(A)(B) = 1 iff $A \cap B = \varnothing$
b. EVERY(A)(B) = 1 iff $A - B = \varnothing$
c. MOST(A)(B) = 1 iff $|A \cap B| > |A|/2$

Here is a representative sample of these three classes (our main concern in what follows).

(4) **Some intersective Dets**
cardinal some, a/an, no, practically no, several, between six and ten, infinitely many, more than six, at least/exactly/just/only/fewer than/at most six, between six and ten, just finitely many, about/nearly/approximately a hundred, a couple of, a dozen, How many?
non-cardinal Which?, more male than female, no...but John (as in *No student but John came to the party*)

(5) **Some co-intersective Dets**
co-cardinal every/all/each, almost all, all but six, all but at most six, all but finitely many
non-co-cardinal every...but John

(6) **Some properly proportional Dets** (proportional but not intersective or co-intersective).
a. more than half the, less than two thirds, less than/at most/at least/exactly half, at most ten per cent, between a half and two thirds, between ten and twenty per cent, all but a tenth, almost a third, What percentage?
b. most, every third (as in *Every third student was inoculated*), just/nearly/exactly/only/not one...in ten (as in *Just one student in ten was inoculated*), almost/more than/less than/exactly seven out of ten (as in *Seven out of ten sailors smoke Players*), between six

out of ten and nine out of ten

So the proportionality Dets include mundane fractional and percentage expressions, (6a), usually built on a *partitive* pattern with *of* followed by a definite DP, as in *most of the students, a third of John's students, ten per cent of those students*, etc. (*half* is slightly exceptional, only taking *of* optionally: *half the students, half of the students* are both fine). The precise syntactic analysis of partitive constructions is problematic in the literature. Keenan & Stavi (1986) treat *more than a third of the* as a complex Det. But more usually linguists treat the expression following *of* as a definite DP and *of* expresses a partitive relation between that DP and the Det that precedes *of*. Barwise and Cooper (1981) provide a compositional semantics for this latter approach which we assume here.

Proportionality Dets also include those in (6b) which are not partitive, but are followed directly by the Noun as in the case of intersective and co-intersective Dets.

DPs built from proportionality Dets usually require that their Noun argument denotes a finite set. We could invent a meaning for *a third of the natural numbers* but this would be a creative step, extending natural usage not simply an act of modeling ordinary usage. In general the functions denoted by proportionality Dets are not intersective or co-intersective, though a few extremal cases are: *Exactly zero per cent = no, a hundred percent = every, more than zero per cent = some, less than a hundred per cent = not every.*

Complex members in each of these seven classes can be formed by taking boolean compounds in *and* (*but*), *or*, *not*, and *neither...nor*. We note without proof:

Proposition 1

a. $GQ_E = [P(E) \rightarrow \{0,1\}]$ is a (complete atomic) boolean algebra inheriting its structure pointwise from $\{0,1\}$.

b. Each of the classes K defined in Def 1 is closed under the pointwise boolean operations and is thus a (complete, atomic) boolean subalgebra of $[P(E) \rightarrow GQ_E]$.

So if D of type <1,1> is intersective (cardinal, co-intersective,...) so is $\neg D$, which maps each A to $\neg(D(A))$, the complement of the GQ D(A). Thus boolean compounds of expressions in any of these classes also lie in that class. E.g. *at least two and not more than ten* is cardinal because *at least two* and *more than ten* are, etc.

We write INT_E ($CARD_E$,...) for the set of intersective (cardinal, ...) functions of type <1,1> over a domain E, omitting the subscript $_E$ when no confusion results. Many relations between these subclasses of Dets are known. E.g. INT, CO-INT, PROP are all subsets of CONS; CARD and CO-CARD are PI subsets of INT and CO-INT respectively. When E is finite CARD = INT∩PI and CO-CARD = CO-INT∩PI. And an easily shown fact, used later, is:

Proposition 2 $INT_E \cap CO\text{-}INT_E = \{\mathbf{0}, \mathbf{1}\}$, where **0** is that constant function of type <1,1> mapping all A,B to 0; **1** maps all A,B to 1.

proof One checks easily that **0** and **1** are both intersective and co-intersective. For the other direction let $D \in INT_E \cap CO\text{-}INT_E$. Then for A,B arbitrary, DAB = D(A∩B)(E), since D is intersective, = D(∅)(E) since D is co-intersective. Thus D is constant, so D = **0** or D = **1**. □

Thus only the two trivial Det functions are both intersective and co-intersective. Further

Fact 1 In general Dets are not ambiguous according as their denotations are intersective or co-intersective.

fewer than zero denotes **0** which is both intersective and co-intersective, but Fact 1 says that no Det expression has *two* denotations, one intersective and the other co-intersective.

Proportionality Dets We begin with some basic facts regarding proportionality Dets:

Not first order definable Barwise and Cooper (1981) argue that MOST as defined here, is not definable in first order logic. See also Westerståhl (1989). The arguments given in these two sources extend to the non-trivial proportionality Dets – those which are not also intersective or co-intersective. Given that the proportionality Dets in general are not first order definable (FOD) it is unsurprising that we have little understanding of their inferential behavior as inference patterns have been best studied for first order expressions.

Not sortally reducible We say that a possible Det function D is *sortally reducible* iff there is a two place boolean function h such that for all subsets A,B of E, DAB = D(E)(h(A,B)). Note that intersective D and co-intersective D are sortally reducible, as illustrated below with **some** and **all**:

(8)
a. Some poets are socialists
b. Some individuals are both poets and socialists

c. All poets are socialists
d. All individuals are either not poets or are socialists
(≡ All individuals are such that if they are poets then they are socialists)

In fact Keenan (1993) shows that the conservative D which are sortally reducible are just the intersective and co-intersective ones. Most reasoning techniques used with formulas of the form $\exists x\phi$ or $\forall x\phi$ involve removing the quantifiers, reasoning with the resulting formula, and then restoring the quantifiers when needed. But such techniques will not apply directly to Ss built with proper proportionality quantifiers as they do not admit of a translation which eliminates the Noun domain of the variable in favor of the entire universe as in (8b) and (8d) above.

Are permutation invariant Given a permutation h of E (so h is a bijection from E to E) we extend h to subsets of E by setting $h(X) = \{h(x)|x \in X\}$, all $X \subseteq E$. And a possible Det denotation D is said to be PI (*permutation invariant*) iff for all permutations h of E,

$$D(A)(B) \quad = \quad D(h(A),h(B))$$

Proportionality Dets (over finite E) always denote PI functions (in distinction for example to **no ... but John** or **Which?** among the intersective Dets).

Have two place variants like intersective Dets, as in:

(9) A greater percentage of teachers than (of) students signed the petition
The same proportion of little boys as (of) little girls laugh at funny faces
Proportionately fewer female students than male students get drafted

(10) (A GREATER PERCENTAGE OF A THAN B)(C) = 1 iff $|A\cap C|/|A| > |B\cap C|/|B|$

Inference paradigms To begin our study of inference paradigms proportionality Dets enter we first review Westerståhl's result concerning our previous attempt (Keenan 2004). That work built on three operations defined on GQs: *complement*, *post-complement*, and *dual*. Complement has already been defined (pointwise) above. For the others:

Def 2 a. $F_{\neg}$, the *postcomplement* of F, is that GQ mapping each B to F($\neg$B), that is, to F(E – B)
b. F^d, the *dual* of F, $=_{def}$ $\neg(F_{\neg})$. Note that $\neg(F_{\neg}) = (\neg F)_{\neg}$, so we may omit parentheses.

We extend these operations pointwise to type <1,1> functions:

Def 3 For D of type <1,1>, $\neg$D, $D_{\neg}$, and D^d are those type <1,1> functions defined by:

a. $\neg$D maps each set A to $\neg$(D(A))
b. $D_{\neg}$ maps each set A to $(D(A))_{\neg}$
c. D^d maps each set A to $(D(A))^d$

Some Examples. We write negX for a DP which denotes the complement of the denotation of X; similarly Xneg denotes its postcomplement and dualX its dual.

X	some	every	more than half	less than half
negX	no	not every	at most half	at least half
Xneg	not every	no	less than half	more than half
dualX	every	some	at least half	at most half

So the complement of *every boy* is *not every boy*, its postcomplement is *no boy*, and its dual is *some boy*. And the complement of *more than half* is *at most half*, its postcomplement is *less than half*, and its dual *at least half*. Observe that the postcomplement and dual operators preserve the property of being proportional but interchange the intersective and co-intersective Dets:

Proposition 3 For D of type <1,1>,
a. if D is proportional so are $D_{\neg}$ and D^d, but
b. if D is intersective (cardinal), $D_{\neg}$ and D^d and both co-intersective (co-cardinal), and
c. if D is co-intersective (co-cardinal), then $D_{\neg}$ and D^d are both intersective (cardinal).

Proof sketch We show b. above, as it plays a role in our later discussion. Let D be intersective. We show that $D_{\neg}$ is co-intersective. Let $A - B = X - Y$. We must show that $D_{\neg}AB = D_{\neg}XY$. But $D_{\neg}AB = (DA)_{\neg}(B) = DA(\neg B) = DA(A \cap \neg B)$, since D is intersective, $= DA(A - B) = D(E)(A \cap (A - B)) = D(E)(A - B) = D(E)(X - Y) = \ldots = D_{\neg}XY$, completing the proof. To see that D^d is co-intersective we observe that $D_{\neg}$ is by the above and so then is $\neg(D_{\neg}) = D^d$ since pointwise complements preserves co-intersectivity (Prop 1). □

Westerståhl's generalization We repeat (1a) above, (1b) being similar.

(1) a. More than n/m of the As are Bs
At least 1 – n/m of the As are Cs
Ergo: Some A is both a B and a C

Now the relevant Dets are interpreted as in (11):

(11) For $0 \leq n \leq m$, $0 < m$,

(MORE THAN n/m)(A)(B) = 1 iff $A \neq \varnothing$ and $|A \cap B|/|A| > n/m$
(LESS THAN 1 – n/m)(A)(B) = 1 iff $A \neq \varnothing$ and $|A \cap B|/|A| < 1 - n/m$
(AT LEAST 1 – n/m)(A)(B) = 1 iff $A \neq \varnothing$ and $|A \cap B|/|A| \geq 1 - n/m$

Westerståhl (pc) notes that the DPs in the premisses in (1a) are duals. (LESS THAN 1 – n/m) is the postcomplement of (MORE THAN n/m) and (AT LEAST 1 – n/m) is its dual.

Westerståhl's Generalization For D conservative, [1] and [2] below are equivalent:

[1] D is right increasing (= increasing on its second argument)
[2] $D(A)(B) \wedge D^d(A)(C) \Longrightarrow \text{SOME}(A)(B \cap C)$

Proof $\Leftarrow$ Let D be conservative and assume [2]. We show [1]. Let $B \subseteq B'$ and assume DAB. We must show DAB'. Assume otherwise. So DAB' = 0. Then $(DA)_{\neg}(\neg B') = 0$, so $\neg(DA)_{\neg}(\neg B') = D^d(A)(\neg B') = 1$. So by [2], $A \cap B \cap \neg B' \neq \varnothing$, contradicting that $B \subseteq B'$. Thus DAB' = 1, and D is right increasing.

$\Rightarrow$ Let D be right increasing and assume DAB = 1 and $D^dAC = 1$, whence by the conservativity of D and D^d we have $DAA \cap B = 1$ and $D^d\, AA \cap C = 1$. Assume leading to a contradiction that $A \cap B \cap C = \varnothing$. Then $A \cap B \subseteq \neg C$, so $D(A)(\neg C) = 1$ by the right increasingness of D. Thus $D_{\neg}(A)(C) = 1$. But $\neg D_{\neg}(A)(C) = D^dAC = 1$, a contradiction. So $A \cap B \cap C \neq \varnothing$, whence SOME(A)($B \cap C$) = 1, establishing [2]. □

So [2] generalizes the argument paradigm in (1a,b) and does not seem specific to proportionality Dets since it holds for DPs built from right increasing conservative Dets in general. So far however I have found it difficult to find examples of non-proportional Dets which instantiate [1] and [2]. One's first guess, *some* and *every*, satisfies [1] and [2] but these Dets are, recall, proportional: SOME = MORE THAN ZERO PER CENT and EVERY = 100%. The only other cases I can think of are ones that make presuppositions on the cardinality of their first argument. Perhaps the least contentious is *both* and *at least one of the two*. *Both students are married* and *At least one of the two students is a vegan* imply *Some student is both married and a vegan*. But this instance does require taking *at least one of the two* as a Det, which we decided against earlier.

The Mid-Point Theorems

We seek now additional inference patterns that proportionality quantifiers naturally enter. Keenan (2004) observes that natural languages present some non-trivial DPs distinct from first order ones which always assign the same truth value to a predicate and its negation, as in (12a,b) and (12c,d). (13) is the general form of the regularity. Proposition 4 is then immediate.

(12) a. Exactly half the students got an A on the exam
b. Exactly half the students didn't get an A on the exam

c. Between a third and two thirds of the students got an A
d. Between a third and two thirds of the students didn't get an A

(13) DP(P1) $\equiv$ DP(not P1)

Proposition 4 The DPs which satisfy (13) are those which denote in FIX($\neg$) = $\{F \in GQ_E | F = F\neg\}$

At issue then is a syntactic question: just which DPs do satisfy (13)? Let us limit ourselves for the moment to ones of the form [Det+N], as we are interested in isolating the role of the Det. And in characterizing that class do the proportionality Dets play any sort of distinguished role? It seems to me that they do, though I can only give a rather informal statement of that role. Still that informal statement at least helps us to understand why many of the natural examples of Dets which denote in FIX($\neg$) are proportional. We begin by generalizing the observation in (12).

Def 4 For p and q fractions with $0 \leq p \leq q \leq 1$,

a. (BETWEEN p AND q)(A)(B) = 1 iff $A \neq \varnothing$ and $p \leq |A \cap B|/|A| \leq q$

b. (MORE THAN p AND LESS THAN q)(A)(B) = 1 iff $A \neq \varnothing$ and $p < |A \cap B|/|A| < q$

Thus (12a) is true iff there is at least one student and at least a third of the students passed and not more than two thirds passed. Dets of the forms in Def 4 are fixed by postcomplement, $\neg$, when the fractions p,q lie between 0 and 1 and sum to 1. The condition that p+q = 1 guarantees that p and q are symmetrically distributed around the midpoint ½. Clearly $p \leq ½$ since $p \leq q$ and p+q = 1. Similarly $½ \leq q$. The distance from ½ to p is ½ – p, and that from ½ to q is q – ½. And ½ – p = q – ½ iff, adding ½ to both sides, 1 – p = q, iff 1 = p+q. And we have:

(14) **Theorem 5 (the Mid-Point Theorem)** Let p,q fractions with $0 \leq p \leq q \leq 1$, p+q = 1. Then

(BETWEEN p AND q) and (MORE THAN p AND LESS THAN q) are both fixed by $\neg$

The theorem (plus pointwise meets) guarantees the logical equivalence of the (a,b) pairs below:

(15) a. Between one sixth and five sixths of the students are happy ≡
b.. Between one sixth and five sixths of the students are not happy

(16) a. More than three out of ten and less than seven out of ten teachers are married
b. More than three out of ten and less than seven out of ten teachers are not married

A variant statement of this theorem using percentages is:

(17) Let $0 \le n \le m \le 100$ with $n+m = 100$. Then

Between n and m per cent of the As are Bs ≡
Between n and m per cent of the As are not Bs

For example choosing $n = 40$ we infer that (18a) and (18b) are logically equivalent:

(18) a. Between 40 and 60 per cent of the students passed
b. Between 40 and 60 per cent of the students didn't pass

And (14) and (17) are (mutual) entailment paradigms which appear to use proportionality Dets in an essential if not completely exclusive way. Many of the pairs of proportional Dets will not satisfy the equivalence in (14) since their fractions do not straddle the mid-point appropriately. And Dets such as *between 10 and 20 per cent* do not satisfy (17) for the same reason. A very large class of complex proportional Dets which satisfy (14) or (17) is given by Theorem 6:

Theorem 6 FIX(⌐) is closed under the pointwise boolean operations.
Proof in the Appendix. □

And given our earlier observation that proportionality functions are closed under the pointwise boolean operations we infer that all the Dets that can be built up as boolean compounds of the basic fractional and percentage Dets in (14) and (17) respectively are both proportional and fixed by ⌐, so they satisfy the equivalence in (13). For example

(19) Either less than a third or else more than two thirds of the As are Bs ≡
Either less than a third or else more than two thirds of the As are not Bs

Proof The Det in this example denotes the boolean complement of BETWEEN A THIRD AND TWO THIRDS and is thus proportional and fixed by ⌐. □

It is perhaps worth noticing what happens with proportional Dets of the form *Between p and q* when their distribution with respect to the mid-point (½, 50%) changes. If both p and q lie below, or both above the midpoint then we have:

Proposition 7 If $0 < p \le q < ½$ or $½ < p \le q < 1$ then

Between p and q of the As are Bs ≡
It is not the case that between p and q of the As are not Bs

Thus such Det pairs satisfy the equivalences in (20).

(20) $D(A)(B) \equiv D^d(A)(B) \equiv (\neg D(A))(\neg B) \equiv \neg(D(A)(\neg B))$

In contrast if the fraction (percentage) pairs p,q include the mid-point but are not centered then no entailment relation in either direction holds. In (21a,b) neither entails the other:

(21) a. Between a third and three quarters of the students passed the exam
b. Between a third and three quarters of the students didn't pass the exam

Generalizing the Mid-Point Theorem

We observe first that the proportionality Dets differ from the intersective and co-intersective ones in being closed under the formation of postcomplements:

Proposition 8 If D of type <1,1> is intersective and $D = D_\neg$ then by Prop (3.b) $D_\neg$ is co-intersective, and since $D_\neg = D$, D is both intersective and co-intersective and hence trivial ($D = \mathbf{0}$ or $D = \mathbf{1}$). Similarly D is trivial if D is co-intersective and $D = D_\neg$. □

Moreover the expression of the postcomplement relation is natural and does not use a distinctive syntax. Here are the simplest cases:

(22)

	POSTCOMPLEMENT
more than n/m	less than 1 – n/m
exactly n/m	exactly 1 – n/m
at most n/m	at least 1 – n/m
more than n %	less than 100 – n%
exactly n%	exactly 100 – n%
at most n%	at least 100 – n%

Notice that in our first group n/m ranges over all fractions and so includes 1 – n/m = (m – n)/m. Similarly in the second group n ranges at least over the natural numbers between 0 and 100 (inclusive) so includes both n% and (100 – n)%. Thus the linguistic means we have for expressing ratios covers proportionality expressions and their postcomplements indifferently. (Note that postcomplement is symmetric: $D = F_\neg$ iff $F = D_\neg$). Recall also that all the natural classes we have adduced are closed under the pointwise boolean operations, expressible with appropriate uses of *and*, *or* and *not.*

Now recall the fractions p,q for which the Mid-Point Theorem holds. If p = n/m and p and q sum to 1 then q = 1 – n/m. And *more than n/m* and *less then 1 – n/m* are postcomplements. (If more than 3/10ths of the As are Bs then less than 7/10ths of the As are non-Bs). Similarly *between n/m and 1 – n/m* just means the same as *at least n/m and at most 1 – n/m*. So we have

Theorem 9 (Generalized Mid-Points) For D of type $\langle 1,1 \rangle$, $(D \wedge D\neg)$ and $(D \vee D\neg)$ are fixed by $\neg$, as are their complements $(\neg D \vee D^d)$ and $(\neg D \wedge D^d)$

partial proof a. $(D \wedge D\neg)\neg = D\neg \wedge D\neg\neg = D\neg \wedge D = D \wedge D\neg$

b. $\neg(D \wedge D\neg) = (\neg D \vee \neg D\neg) = (\neg D \vee D^d)$ and

$(\neg D \vee D^d)\neg = (\neg D\neg \vee D^d\neg) = (D^d \vee \neg D) = (\neg D \vee D^d)$

These proofs use:

Proposition 10 Fix($\neg$) is closed under pointwise complements, meets and joins and is in fact a complete (and thus atomic) subalgebra of GQ_E.

Proposition 11 The postcomplement function $\neg$ is self inverting ($D\neg\neg = D$) and thus bijective, and it commutes with $\neg$ and $\wedge$ thus is a boolean automorphism of GQ_E.

Corollary 12 If D is an expressible proportionality function then so are $(D \wedge D\neg)$ and $(D \vee D\neg)$, and $(\neg D \vee D^d)$ and $(\neg D \wedge D^d)$, and the Dets which express them satisfy (13).

Below we give some (more) examples of proportionality Dets of the form in Cor 12. Colloquial expression may involve turns of phrase other than simple conjunction and disjunction.

(23) **Some examples**

A. more than three tenths but less than seven tenths
more than three out of ten but less than seven out of ten
more than thirty per cent but less than seventy per cent

exactly a quarter or exactly three quarters
exactly one in four or exactly three out of four
exactly twenty-five per cent or exactly seventy-five percent

at least three tenths and at most seven tenths
at least three out of ten and at most seven out of ten
at least thirty per cent and at most seventy per cent

between a quarter and three quarters
between twenty-five per cent and seventy-five per cent

exactly one (student) in ten or exactly nine (students) in ten

B. not more than three tenths or not less than seven tenths
= at most three tenths or at least seven tenths

(27) a. Either all or none of the students will pass that exam $(F \vee F_{\neg})$

b. Either all or none of the students won't pass that exam

Note that the (compound) Dets in these two examples are properly proportional: *some but not all* ≡ *more than zero per cent and less than 100 per cent*, and *all or none* ≡ *either 100 per cent or else exactly zero per cent*. For the record,

(28) *some but not all*, which denotes $(\text{SOME} \wedge \neg\text{ALL})$, is proportional and not intersective or co-intersective (E assumed to have at least two elements).

Basically *some but not all* fails to be intersective because of the *not all* part which is not intersective; and it fails to be co-intersective because *some* fails to be. To see that it is proportional, suppose that the proportion of As that are Bs is the same as the proportion of Xs that are Ys. Then if *Some but not all As are Bs* is true then at least one A is a B and at least one A is not a B, so the percentage of As that are Bs lies strictly between 0% and 100%, which is exactly where the percentage of Xs that are Ys lies, whence *some but not all Xs are Ys*. One sees then that the (complete) boolean closure of $\text{INT}_E \cup \text{CO-INT}_E$ includes many functions that lie outside INT_E and CO-INT_E. In fact, Keenan (1993), this closure is exactly the set of conservative functions and so includes in particular all the conservative proportional ones.

Note now however that (29a,b) are logically equivalent, as the <1,1> functions the Dets denote are postcomplements, but *either exactly five or else all but five* is not proportional:

(29) a. Either exactly five or else all but five students came to the party

b. Either exactly five or else all but five students didn't come to the party

To see this let A have 100 members, just five of which are Bs. The D denoted by the Det in (29a) maps A,B to 1. But for $|X| = 1{,}000$ and $|X \cap Y| = 50$, that D maps X,Y to 0, even though the proportion of Xs that are Ys, 1/20, is the same as the proportion of As that are Bs.

Clearly then certain boolean compounds of intersective with co-intersective Dets yields some non-proportional Dets which satisfy Theorem 9, so that paradigm is not limited to proportionality Dets.

A last case of DPs that may satisfy Theorem 9 is given by partitives of the form in (30):

(30) a. (EXACTLY n OF THE 2n)(A)(B) = 1 iff $|A| = 2n$ and $|A \cap B| = n$

b. (BETWEEN n and 2n of the 3n)(A)(B) = 1 iff $|A| = 3n$ and $n \leq |A \cap B| \leq 2n$ $(n > 0)$

Of course in general DPs of the form *exactly n of the m Ns* are not fixed by $\neg$. But in the case where $m = 2n$ they are. Note that if we treat *exactly n of the m* as a Det (an analysis that we, along with most linguists, reject) we have:

(31) For $m > n$, *exactly n of the m* is in general not intersective, co-intersective or proportional (but is conservative and permutation invariant).

Appendix

Proposition 1

Proof sketch (1b). Let D be conservative, let $A \cap B = A \cap B'$. We show that $(\neg D)(A)(B) = (\neg D)(A)(B')$. $(\neg D)(A)(B) = \ldots = \neg(DAB) = \neg(DA(A \cap B)) = \neg(DA(A \cap B')) = \neg(DAB') = \ldots = (\neg D)(A)(B')$. For $D_{\ulcorner}$, let $A \cap B = A \cap B'$. Then $D_{\ulcorner}AB = DA(\neg B) = DA(A - B) = DA(A - (A \cap B)) = DA(A - (A \cap B')) = DA(A - B') = DA(\neg B') = D_{\ulcorner}(A)(B)$. □

(2c). Let D be intersective. Let $A - B = X - Y$ and show that $D_{\ulcorner}AB = D_{\ulcorner}XY$. $D_{\ulcorner}AB = (DA)_{\ulcorner}(B) = DA(\neg B) = DA(A - B) = D(A - B)(A - B)$, since $A \cap (A - B) = (A - B) \cap (A - B)$, $= D(X - Y)(X - Y) = DX(X - Y) = DX(\neg Y) = (DX)_{\ulcorner}(Y) = D_{\ulcorner}XY$. □ Further, since D is intersective so is $\neg D$ by Prop 1, whence by the above, $(\neg D)_{\ulcorner} = D^d$ is co-intersective. □

The First Mid-Point Theorem

Let $0 \leq p \leq q \leq 1$ with $p+q = 1$. Then

Between p and q of the As are Bs ≡ *Between p and q of the As are not Bs*

proof Assume (BETWEEN *p* AND *q*)(A)(B) = 1. Show (BETWEEN *p* AND *q*)(A)($\neg$B) = 1. Suppose leading to a contradiction that $|A \cap \neg B|/|A| < p$. Then the percentage of As that are Bs is greater than q, contrary to assumption. The second case in which $|A \cap \neg B|/|A| > q$ is similar, hence the percentage of As that aren't Bs lies between p and q. □

Theorem 6 FIX($\ulcorner$) is closed under the pointwise boolean operations.

a. Let $D \in$ FIX($\ulcorner$). We must show that for all sets A, $(\neg D)(A) = ((\neg D)(A))_{\ulcorner}$, that is, $\neg D$ is fixed by $\ulcorner$. Let A, B arbitrary. Then

$(\neg D)(A)(B)$	=	$\neg(D(A)(B))$	Pointwise $\neg$ (twice)
	=	$\neg((D(A))_{\ulcorner}(B))$	D(A) is fixed by $\ulcorner$
	=	$\neg(D(A))_{\ulcorner}(B)$	Pointwise $\neg$
	=	$((\neg D)(A))_{\ulcorner}(B)$	Pointwise $\neg$

Thus $(\neg D)(A) = ((\neg D)(A))_{\ulcorner}$, as was to be shown.

b. Show $(D \wedge D') = (D \wedge D')_{\ulcorner}$, i.e. show $(D \wedge D')(A) = ((D \wedge D')(A))_{\ulcorner}$

$$
\begin{aligned}
(D \wedge D')(A)(B) &= (DA \wedge D'A)(B) \\
&= ((DA)_{\ulcorner} \wedge (D'A)_{\ulcorner})(B) \\
&= (DA)_{\ulcorner}(B) \wedge (D'A)_{\ulcorner}(B) \\
&= DA(\neg B) \wedge D'A(\neg B) \\
&= (DA \wedge D'A)(\neg B) \\
&= (D \wedge D')(A)(\neg B) \\
&= (D \wedge D')(A)_{\ulcorner}(B)
\end{aligned}
$$

Essentially the same proof carries over for $\bigwedge_i D_i$ replacing $D \wedge D'$ showing completeness.

Atomicity then follows. □

References

Barwise, J. and R. Cooper. 1981. Generalized quantifiers and natural language. In *Linguistics and Philosophy* vol 4: 159 – 219.

Keenan, E.L. 1993. Natural language, sortal reducibility and generalized quantifiers. In *The Journal of Symbolic Logic* vol 58:1: 314 – 325.

Keenan, E.L. 2004. Excursions in natural logic. In *Language and Grammar: Studies in Mathematical Linguistics and Natural Language*. C. Casadio, P.J. Scott, and R.A.G. Seely (eds) CSLI. Stanford

Keenan, E.L. and J. Stavi. 1986. A semantic characterization of natural language determiners. In *Linguistics and Philosophy* vol 9: 253 – 326.

Lindstrom, P. 1966. First-order predicate logic with generalized quantifiers. *Theoria* 35:186 – 195.

Westerstahl, D. 1989. Quantifiers in formal and natural languages. In *Handbook of Philosophical Logic* vol IV. D. Gabbay and F. Guenthner eds. Reidel, Dordrecht. Pp. 1 – 131.